Denis Bonheure

A Study of Heteroclinic Orbits for a Class of Fourth Order Ordinary Differential Equations

Denis Bonheure
Institut de mathématique pure et appliquée
Université catholique de Louvain
Chemin du Cyclotron, 2
1348 Louvain-la-neuve (Belgique)

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Jury: Christian Fabry, Promoteur, UCL
Jean-Pierre Gossez, Université Libre de Bruxelles, Belgique
Patrick Habets, Promoteur, UCL
Jean Mawhin, Président du jury, UCL
Luís Sanchez, Universidade de Lisboa, Portugal
Charles Stuart, École Polytechnique Fédérale Lausanne, Suisse
Michel Willem, UCL

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duc@ciaco.com

À Ana

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Préface

Durant ces cinq dernières années passées au sein du groupe d'analyse non linéaire du Département de mathématique de l'Université Catholique de Louvain, j'ai eu l'occasion d'étudier divers aspects de la théorie des équations différentielles ordinaires (EDOs) et des équations aux dérivées partielles (EDPs). Je me suis plus particulièrement intéressé aux EDOs bien que j'aie récemment étudié quelques EDPs elliptiques. Les différents problèmes que j'ai traités et les différentes personnes avec qui j'ai travaillé m'ont donné l'opportunité de découvrir plusieurs méthodes classiques telles que la *méthode des sur- et sous-solutions*, la *théorie du degré topologique* ou encore les *méthodes variationnelles.*

Étant donné que mes recherches ont couvert des sujets fort différents, j'ai choisi de me concentrer dans cette dissertation sur un des problèmes qui m'a principalement intéressé : la recherche de *solutions hétéroclines* pour des EDOs bi-stables du quatrième ordre. Le choix de ce sujet a été fortement influencé par mes derniers travaux : j'ai rédigé conjointement avec Luís Sanchez, Professeur à l'Université de Lisbonne, un panorama des résultats récents concernant les solutions hétéroclines pour des classes de systèmes d'EDOs du second ordre et d'EDOs scalaires du quatrième ordre [**26**]. Cette monographie constitue un chapitre du "Handbook of Differential Equations, Volume 2" à paraître aux éditions Elsevier et édité par A. Cañada, P. Drabek et A. Fonda.

Cependant, j'ai aussi voulu m'arrêter brièvement sur les autres problèmes qui ont retenu mon attention pendant mes années de doctorat. Je consacre à cet effet un *Avant-propos*, indépendant du reste de cette dissertation.

Remerciements. La liste des gens qui m'ont aidé ou soutenu durant ces cinq dernières années est assez longue et je ne pourrais citer tout le monde.

J'aimerais tout d'abord remercier Christian Fabry et Patrick Habets qui m'ont appris énormément et avec qui j'ai travaillé avec beaucoup de plaisir. Luís Sanchez mériterait très certainement, lui aussi, le titre de "orientador" comme on dit au Portugal. Il s'est toujours montré très disponible et enthousiaste lors de mes visites à Lisbonne et je l'en remercie très chaleureusement.

Mon arrivée à Louvain-la-Neuve en 1999 n'aurait sans doute pas été possible sans Jean-Pierre Gossez et Patrick Habets. Par ailleurs, je remercie le Département d'ingénierie mathématique de la Faculté des sciences appliquées de m'avoir accordé sa confiance.

J'aimerais aussi remercier tous les membres du Département de mathématique qui m'ont très vite adopté. Je me suis senti rapidement "chez moi" grâce notamment à la bonne ambiance générale et à l'amitié de Marielle, Paul, Yves, Paolo, Marino et Isabella. Je n'oublie pas non plus l'équipe d'analyse non linéaire (que j'élargi bien entendu à l'ULB) et l'école doctorale de mathématique qui m'ont offert une structure idéale pour poursuivre mes recherches ; merci à Yves Felix, Christian Fabry, Jean-Pierre Gossez, Patrick Habets, Enrique Lami Dozo, Jean Mawhin, Michel Willem et aux nombreuses personnes que j'ai croisées lors des séminaires à Louvain-la-Neuve ou à Bruxelles.

Mes séjours à l'étranger ont souvent été un moteur dans mes recherches. Le Centro de Matemática e Aplicações Fundamentais à Lisbonne est assurément un lieu où je me sens bien. Je tiens à remercier tous ceux qui m'y ont accueilli, à chaque fois, très amicalement. Je suis très heureux d'avoir eu l'occasion de collaborer avec Luís et Zé Maria, j'espère que l'avenir nous réunira encore souvent. Colette De Coster m'a reçu chaleureusement à l'Université du Littoral, Milan Tvrdý et Stefan Schwabik au Mathematical Institute of the Academy of Sciences of the Czech Republic et Massimo Tarallo m'a réservé un accueil particulièrement enthousiaste à l'Università degli Studi di Milano. Merci aussi à Enrico Serra.

Il est temps de passer aux personnes qui m'ont assisté dans l'écriture de cette thèse : Patrick Habets a eu le courage de relire entièrement mon manuscript et m'a donné de précieux conseils ; Christian Fabry m'a permis d'améliorer la rédaction de l'avant-propos ; Ana, Romina, David et Serge ont chacun relu des morceaux de texte et éliminé un nombre important de fautes de frappe et d'orthographe. Ils sont d'ores et déjà pardonnés pour celles qui ont échappé à leurs yeux !

Je terminerai par des remerciements plus personnels : tout d'abord à ma famille qui m'a toujours soutenu, ensuite à Ana qui a supporté mon stress, mes absences et qui m'apporte du bonheur au quotidien.

Louvain-La-Neuve, Novembre 2004

Avant-propos

Cette section est indépendante du reste du texte. J'y présente sans trop de détails les autres problèmes sur lesquels j'ai travaillé ces cinq dernières années. J'ai sélectionné les résultats principaux, parfois sous une forme simplifiée. Mon premier souci est de donner une idée aussi précise que possible de mes travaux sans entrer dans des considérations trop techniques. Les résultats sont livrés sans démonstrations formelles pour lesquelles je renvoie le lecteur aux articles originaux. Je présente un aperçu des méthodes que j'ai utilisées ainsi que les arguments centraux.

A ma connaissance, les travaux que je présente sont nouveaux et complètent de quelques minuscules pièces un immense puzzle commencé depuis des siècles. Je mentionne les pièces adjacentes auxquelles je me suis arrimé et je les compare quand il y a lieu. L'ensemble de mes résultats est le produit de fructueuses collaborations. J'ai bénéficié de l'expérience et de l'enthousiasme de Christian Fabry, Luís Sanchez, Colette De Coster et Patrick Habets ainsi que de la vivacité de jeunes collègues José Maria Gomes, David Ruiz et Didier Smets.

Cet avant-propos est découpé en six sections principales pouvant être lues séparément, bien que les trois premières soient très étroitement liées. La quatrième section peut se rattacher aux précédentes sans que la lecture de celles-ci soit indispensable. Les deux derniers problèmes que j'ai traités sont totalement indépendants. La succession des sujets respecte l'ordre chronologique dans lequel je les ai abordés. Elle témoigne également de mon intérêt croissant pour les problèmes variationnels d'autant qu'il faut insérer dans cette liste le sujet principal de cette thèse qui repose sur des méthodes de minimisation. Néanmoins, cela ne signifie pas que je renonce aux problèmes sans structure variationnelle et que j'abandonne les méthodes avec lesquelles j'ai fait mes premières armes. En effet, j'ai tout d'abord découvert les sur- et sous-solutions que j'ai utilisées dans mon premier travail avec Colette De Coster. Elles m'ont ensuite mené vers le degré topologique sur lequel repose les résultats obtenus avec Christian Fabry et Didier Smets. Dans mes travaux suivants, Christian Fabry m'a fait découvrir différentes approches que l'on pourrait qualifier de "plus directes" car elles ne reposent pas directement sur des résultats abstraits de nature topologique ou autre. Sous l'impulsion de David Ruiz, j'ai fait un saut vers le monde variationnel. Dans le même temps, Luís Sanchez m'introduisait aux équations du quatrième ordre et aux méthodes de minimisation dans la recherche de solutions hétéroclines. Dans mes deux derniers travaux, je me suis orienté vers les équations aux dérivées partielles tout en confirmant mon goût pour les méthodes variationnelles.

Cette section contient une bibliographie qui lui est propre ainsi qu'un index des définitions et notations introduites.

Problèmes singuliers

Le premier problème que j'ai étudié concerne des équations singulières du second ordre dont l'équation

$$u'' + \frac{a}{u^3} + bu = h(t) \tag{1}$$

est un modèle simple. Les paramètres a et b sont des nombres réels et la fonction h, appelée *force extérieure* ou *terme forçant* , est continue et périodique. La force

$$f(u) = -\frac{a}{u^3} - bu$$

est singulière à l'origine, répulsive si a est négatif et attractive si a est positif. De même, f est répulsive à l'infini si b est positif, attractive si b est négatif. La question principale qui nous intéresse concernant ce modèle est celle de l'existence de solutions périodiques de même période T que la force extérieure. Sans aucune perte de généralité, nous fixons dès à présent $T = 2\pi$. Il est bien connu que lorsque le problème est linéaire, c'est-à-dire lorsque $a = 0$, l'existence de solutions 2π-périodiques dépend de la valeur de b et de la fonction h. C'est l'alternative de Fredholm. Si $b \neq n^2$ pour tout $n \in \mathbb{N}$, alors l'équation (1) avec $a = 0$ possède une (et une seule) solution 2π-périodique quelle que soit la fonction $h \in C_{2\pi}(\mathbb{R}) := \{u \in C(\mathbb{R}) \mid u(t+2\pi) = u(t),\ t \in \mathbb{R}\}$. Les valeurs $\lambda_n := n^2$, $n \in \mathbb{N}$, sont les valeurs propres de l'opérateur $Lu := -u''$ défini sur l'espace fonctionnel $C^2_{2\pi}(\mathbb{R}) := \{u \in C^2(\mathbb{R}) \mid u(t+2\pi) = u(t),\ t \in \mathbb{R}\}$ et à valeurs dans $C_{2\pi}(\mathbb{R})$. Lorsque b est une de ces valeurs propres, le problème périodique associé à l'équation (1) est dit *résonant.* La notion de résonance peut s'interpréter de diverses manières. Je l'associerai simplement au fait que l'équation (1) sans terme forçant (c.-à-d. avec $h = 0$) possède un ensemble non borné de solutions 2π-périodiques, bien que d'autres aspects des problèmes résonants seront abordés plus loin dans mon exposé. Une équation non forcée est appelée *autonome* et une solution périodique d'une équation autonome est souvent appelée une *oscillation libre*. Dans le cas de l'équation linéaire, lorsque $b = \lambda_n$ pour un certain naturel n, le sous-espace vectoriel

$$\Lambda_n := \{A \sin nt + B \cos nt \mid A,\ B \in \mathbb{R}\}$$

constitue un ensemble non borné d'oscillations libres de période 2π. Pour le problème forcé, l'existence d'une solution 2π-périodique dépend alors de la fonction h. Si la projection, au sens du produit scalaire canonique de $L^2(0, 2\pi)$, de h dans Λ_n est nulle alors l'équation possède une infinité de solutions 2π-périodiques. Dans le cas contraire, l'équation n'a aucune solution de période 2π. Cette dernière propriété mène à une autre interprétation de la résonance : le problème 2π-périodique associé à l'équation (1) est dit *résonant* si pour une certaine force extérieure h l'équation ne possède aucune solution 2π-périodique. Dans le cas linéaire, cette définition est équivalente à celle précitée.

Considérons à présent le problème singulier ($a \neq 0$). Le cas le plus intéressant est celui d'une singularité répulsive à l'origine et à l'infini, c.-à-d. $a < 0$ et $b > 0$. Pour fixer les idées, nous étudions l'existence de solutions positives. Nous pouvons également poser $a = -1$ sans perte de généralité puisque le cas général peut se traiter par un changement d'échelle. Pour étudier l'équation autonome

$$u'' - \frac{1}{u^3} + bu = 0, \tag{2}$$

nous pouvons utiliser l'intégrale première d'énergie et la représentation des solutions dans le plan des phases (u, u'). Les solutions de l'équation (2) sont toutes périodiques, comme l'indique la Figure 1. Un calcul classique du temps de parcours des orbites en fonction de l'énergie montre que le système est isochrone, ce qui signifie que toutes les solutions ont la même période. Exceptionnellement, nous connaissons explicitement l'ensemble des solutions de l'équation (2) :

$$\mathcal{A} := \left\{ u_A(t) := \sqrt{A^2 \cos^2(\sqrt{b}t) + \frac{\sin^2(\sqrt{b}t)}{A^2 b}} \mid A \in \mathbb{R} \right\}. \tag{3}$$

Ces solutions sont de période $\pi/\sqrt{b}$. On en déduit donc que le problème 2π-périodique associé à l'équation

$$u'' - \frac{1}{u^3} + bu = h(t) \tag{4}$$

est résonant pour les valeurs

$$b \in \Sigma := \left\{ \sigma_n := \frac{n^2}{4} \mid n \in \mathbb{N}_0 \right\},$$

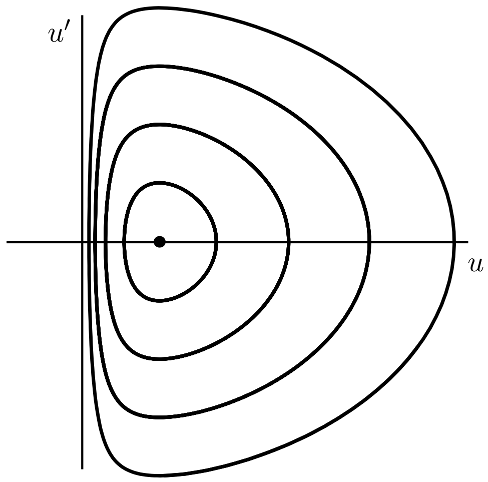

Figure 1. *Plan de phases de l'oscillateur singulier.*

puisque $\mathcal{A}$ est alors un ensemble non borné d'oscillations libres de période 2π. Les valeurs du paramètre pour lesquelles le problème est résonant ne correspondent pas aux valeurs propres λ_n du problème périodique linéaire mais bien aux valeurs propres de l'opérateur L défini sur l'espace fonctionnel $C_0^2([0, 2\pi]) := \{u \in C^2([0, 2\pi]) \mid u(0) = u(2\pi) = 0\}$, c.-à-d. celles associées au problème de Dirichlet. Ceci peut s'expliquer intuitivement en observant que lorsque A tend vers $+\infty$, l'oscillation libre normalisée converge dans L^∞ vers une fonction positive qui, entre chaque pair de zéros consécutifs, est solution d'un problème de Dirichlet. En effet, nous avons

$$\frac{u_A}{|A|} \xrightarrow{L^\infty} u_\infty,$$

où $u_\infty(t) := |\cos(\sqrt{b}t)|$, et tant que $\cos(\sqrt{b}t) \neq 0$, u_∞ est solution du problème linéaire

$$u'' + bu = 0.$$

En réalité, cela signifie que u_∞ est solution d'un problème périodique avec obstacle en zéro. Nous nous attarderons ultérieurement sur ce problème.

Une première question surgit naturellement : existe-t-il un résultat semblable à l'alternative de Fredholm s'appliquant à l'équation (4) ? M. Del Pino, R. Manasevich et A. Montero [DMM] ont démontré que lorsque $b \neq \sigma_n$, l'équation forcée (4) admet une solution quel que soit $h \in C_{2\pi}(\mathbb{R})$. Leur résultat est en fait plus général, il s'applique à des équations du type

$$u'' + f(t, u) = 0, \tag{5}$$

où $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ est une fonction continue telle que $f(t,u) \sim -1/u^\nu$ pour u proche de zéro et $\nu \geq 1$, et

$$\sigma_n < \liminf_{u\to+\infty} \frac{f(t,u)}{u} \leq \limsup_{u\to+\infty} \frac{f(t,u)}{u} < \sigma_{n+1}, \tag{6}$$

pour tout $n \in \mathbb{N}$.

A. C. Lazer et S. Solimini [LaS] avait auparavant étudié une classe de problèmes singuliers incluant le modèle (4) pour $b = 0$ et prouvé l'existence d'une solution 2π-périodique pour chaque force extérieure h de moyenne négative, c.-à-d. telle que

$$\int_0^{2\pi} h(t)\,dt < 0.$$

Par ailleurs, il est évident que dans ce cas, cette dernière condition est nécessaire (il suffit d'intégrer l'équation). D'autres auteurs comme P. Omari et W. Ye [OY] ou A. Fonda [Fo] ont considéré l'équation (5) avec des variantes de l'hypothèse (6) pour $n = 0$.

Différentes méthodes ont été utilisées pour traiter ces problèmes singuliers. A. C. Lazer et S. Solimini, tout comme P. Omari et W. Ye, ont appliqué la méthode des sur- et sous-solutions à un problème tronqué proche de l'origine ; M. Del Pino et al. ont étudié un problème de point fixe équivalent à l'aide du degré topologique de Leray-Schauder ; A. Fonda a traité partiellement le problème de manière variationnelle.

Dans un article en collaboration avec C. De Coster [BDC], nous avons principalement généralisé le résultat de P. Omari et W. Ye pour des équations contenant un terme supplémentaire de friction. Notre résultat étend également l'approche de P. Habets et L. Sanchez [HS]. Le résultat principal concerne le problème aux limites

$$\begin{gathered} u'' + g(u)u' + f(t,u) = h(t), \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). \end{gathered} \tag{7}$$

Avant d'énoncer le théorème, rappelons la notion de sur- et sous-solutions. Dans cette définition, l'espace $W^{2,1}(0,2\pi)$ est l'ensemble des fonctions de classe C^1 dans l'intervalle $[0,2\pi]$ dont la dérivée faible seconde est intégrable.

Définitions 1. Une fonction $\alpha \in W^{2,1}(0,2\pi)$ telle que $\alpha(0) = \alpha(2\pi)$ et $\alpha'(0) \geq \alpha'(2\pi)$ est une *sous-solution* de (7) si pour presque tout $t \in [0,2\pi]$,

$$\alpha''(t) + g(\alpha(t))\alpha'(t) + f(t,\alpha(t)) \geq h(t).$$

Une fonction $\beta \in W^{2,1}(0,2\pi)$ telle que $\beta(0) = \beta(2\pi)$ et $\beta'(0) \leq \beta'(2\pi)$ est une *sur-solution* de (7) si pour presque tout $t \in [0,2\pi]$,

$$\beta''(t) + g(\beta(t))\beta'(t) + f(t,\beta(t)) \leq h(t).$$

Rappelons aussi la définition de fonction de Carathéodory.

DÉFINITIONS 2. Soit $D \subset \mathbb{R}$. Une fonction $f : [0, 2\pi] \times D \to \mathbb{R}$ est dite de *Carathéodory* si
(i) pour presque tout $t \in [0, 2\pi]$, la fonction $f(t, \cdot)$ définie sur l'ensemble D est continue ;
(ii) pour tout $u \in \mathbb{R}$, la fonction $f(\cdot, u)$ définie sur $[0, 2\pi]$ est mesurable.

Si de plus, pour tout ensemble compact $C \subset D$, il existe une fonction $k \in L^1(0, 2\pi)$ telle que, pour presque tout $t \in [0, 2\pi]$ et tout $u \in C$,

$$|f(t, u)| \leq k(t),$$

alors f est dite L^1-*Carathéodory*.

THÉORÈME 1. *Soient $g \in \mathcal{C}(\mathbb{R}^+)$, $f : [0, 2\pi] \times \mathbb{R}_0^+ \to \mathbb{R}$ une fonction L^1-Carathéodory et $h \in L^1(0, 2\pi)$. Supposons de plus que*
(a) *il existe une paire de sur- et sous-solutions de* (7)*, α et $\beta \in \mathcal{C}([0, 2\pi])$, telle que $0 < \alpha$, $0 < \beta$ dans $[0, 2\pi]$ et $\alpha \nleq \beta$;*
(b) (*Singularité forte) il existe $\rho > 0$, $\ell \in L^1(0, 2\pi)$ et $\hat{f} \in \mathcal{C}(]0, \rho])$ avec*

$$\int_0^\rho \hat{f}^-(u)\, du = +\infty \qquad et \qquad \hat{f}^-(u) = \max\{-\hat{f}(u), 0\},$$

tels que pour tout $u \in]0, \rho]$ et presque tout $t \in [0, 2\pi]$

$$f(t, u) \leq \hat{f}(u) \qquad et \qquad f(t, u) \leq \ell(t);$$

(c) *il existe une fonction $\gamma \in L^1(0, 2\pi)$ telle que $\gamma(t) \leq \frac{1}{4}$ presque partout dans $[0, 2\pi]$ avec inégalité stricte sur un ensemble de mesure positive et*

$$\limsup_{u \to +\infty} \frac{f(t, u)}{u} \leq \gamma(t),$$

uniformément en $t \in [0, 2\pi]$.
Alors le problème (7) *possède au moins une solution positive.*

Quelques commentaires s'imposent. L'hypothèse (a) peut paraître un peu artificielle. Elle se vérifie néanmoins dans diverses situations. Pour obtenir une sur-solution, nous pouvons par exemple imposer l'existence d'une constante positive β telle que pour presque tout $t \in [0, 2\pi]$, $f(t, \beta) - h(t) \leq 0$. Si h est une fonction bornée inférieurement, ceci est automatiquement vérifié puisque l'hypothèse (b) implique que

$$\liminf_{u \to 0} f(t, u) = -\infty$$

pour presque tout $t \in [0, 2\pi]$. Quant à l'existence d'une sous-solution, elle se déduit par exemple en supposant qu'il existe $R > \beta$ et $f_0 \in L^1(0, 2\pi)$ tels que, pour presque tout $t \in [0, 2\pi]$, pour tout $u \geq R$, $f(t, u) \geq f_0(t)$ et

$$\int_0^{2\pi} f_0(t)\, dt \geq \int_0^{2\pi} h(t)\, dt.$$

Si f satisfait

$$\limsup_{u\to+\infty} f(t,u) = +\infty$$

pour presque tout t, cette condition est également remplie lorsque h est bornée supérieurement. D'autres alternatives pour obtenir des sur- et sous-solutions sont proposées dans D. Bonheure et C. De Coster [BDC].

L'hypothèse (b) permet d'obtenir une borne a priori inférieure sur les solutions. Le fait de supposer que la singularité est forte implique qu'une solution aurait besoin d'une "énergie" infinie pour toucher la singularité. Or l'hypothèse (c) suffit pour montrer que les solutions sont a priori bornées supérieurement ce qui implique une borne a priori sur l'énergie. Il est important de remarquer qu'un contrôle sur la singularité tel que l'hypothèse (b) est nécessaire. En effet, A. C. Lazer et S. Solimini [LaS] ont démontré qu'il existe une fonction négative $h \in \mathcal{C}([0,2\pi])$ telle que l'équation

$$u'' - \frac{1}{\sqrt{u}} = h(t)$$

n'a pas de solution 2π-périodique. L'hypothèse (c) est une condition de non-résonance. Elle s'avère, elle aussi, indispensable comme nous le verrons ultérieurement.

La démonstration du Théorème 1 repose sur l'étude d'un problème modifié "effaçant" la singularité. En utilisant la paire de sur- et sous-solutions (mal ordonnées), nous démontrons l'existence d'une solution du problème modifié et de bornes a priori pour cette solution. Le point sensible de cette approche consiste à s'assurer que les bornes a priori ne dépendent pas de la modification. Dans ce cas, lorsque la troncature est faite en dessous de la borne a priori inférieure, la solution du problème modifié satisfait en fait l'équation originale.

Dans notre contribution [BDC], nous traitons également la possibilité d'une résonance avec la "valeur critique" $\sigma_1 = 1/4$. Nous nous limitons cette fois au problème sans frottement

$$\begin{gathered} u'' + f(t,u) = h(t), \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). \end{gathered} \tag{8}$$

THÉORÈME 2. *Soient* $h \in L^1(0,2\pi)$ *et* $f : [0,2\pi] \times \mathbb{R}_0^+ \to \mathbb{R}$ *une fonction* L^1*-Carathéodory. Supposons de plus que*

(a) *pour tout* $u > 0$ *et presque tout* $t \in [0,2\pi]$,

$$f(t,u) \le \frac{u}{4}\,;$$

(b) *il existe* $R > 0$ *et une fonction* $f_0 \in L^1(0,2\pi)$ *tels que, pour presque tout* $t \in [0,2\pi]$, *pour tout* $u \ge R$,

$$f(t,u) \ge f_0(t)$$

et

$$\int_0^{2\pi} f_0(t)\,dt \geq \int_0^{2\pi} h(t)\,dt;$$

(c) *il existe* $\delta > 0$ *tel que*

$$\min_{t\in[0,2\pi]} \int_t^{t+2\pi} h(s)\sin\left(\frac{s-t}{2}\right)\,ds \geq \delta.$$

Alors le problème (8) *a au moins une solution positive.*

Ce résultat généralise la condition de non-résonance

$$\inf_{t\in[0,2\pi]} h(t) > 0$$

formulée par I. Rachunková et al. [RTV]. En effet, si $h(t) = a + b\cos t$, l'hypothèse (c) revient à s'assurer que la fonction $4a - \frac{4}{3}b\cos t$ est positive. Cette condition est clairement satisfaite pour des valeurs de a et b telles que la fonction h elle-même change de signe.

Il est intéressant d'observer qu'aucune hypothèse n'est faite sur le comportement de f à l'origine. En particulier, f peut avoir une singularité faible (c.-à-d. ne pas satisfaire l'hypothèse (b) du Théorème 1). C'est l'hypothèse (c) qui permet de se soustraire d'une singularité forte. En effet, cette hypothèse livre immédiatement une borne a priori inférieure pour la famille de problèmes modifiés.

L'hypothèse (c) du théorème précédent peut être reformulée de la manière suivante. Définissons $\psi_1(t) := |\cos\left(\frac{t}{2}\right)|$ et

$$\Phi_1(\theta) := \int_0^{2\pi} h(t)\psi_1(t+\theta)\,dt.$$

La fonction h satisfait dès lors l'hypothèse (c) si Φ_1 est une fonction positive.

Considérons à nouveau l'équation modèle (4). Le théorème précédent permet de déduire l'existence d'une solution 2π-périodique dans le cas résonant $b = 1/4$ si h est tel que la fonction Φ_1 est positive. Rappelons que dans le cas de l'équation (4), la fonction ψ_1 est la limite normalisée des oscillations libres.

Perturbations bornées d'oscillateurs isochrones

L'oscillateur asymétrique. Il est temps de faire une analogie avec une équation non singulière qui, ces dernières années, a beaucoup attiré l'attention. Il s'agit de l'oscillateur asymétrique. C'est un oscillateur dont les coefficients des forces de rappel diffèrent à gauche et à droite de l'origine :

$$u'' + \alpha u^+ - \beta u^- = 0, \tag{9}$$

où α, β sont deux paramètres strictement positifs, $u^+ = \max(u, 0)$ et $u^- = \max(-u, 0)$. Tout comme dans le cas de l'oscillateur singulier (2), une étude élémentaire de l'oscillateur asymétrique montre que toutes les solutions sont périodiques de même période $\pi/\sqrt{\sigma}$, où σ est tel que

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{1}{\sqrt{\sigma}}.$$

Dès lors, si $\sigma = n^2/4$, le problème 2π-périodique associé à l'équation

$$u'' + \alpha u^+ - \beta u^- = h(t), \tag{10}$$

où $h \in C_{2\pi}(\mathbb{R})$, est résonant. Remarquons que dans les cas dégénérés $\alpha = 0$ ou $\beta = 0$, le problème périodique associé à l'équation (10) est également résonant mais l'équation autonome est clairement non isochrone. L'ensemble

$$C_0 := \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha = 0 \text{ ou } \beta = 0\}$$

et l'union des courbes

$$C_n := \{(\alpha, \beta) \in \mathbb{R}^2 \mid \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}\} \quad (n \in \mathbb{N})$$

représentent toutes les valeurs des paramètres (α, β) telles que l'équation autonome (9) soit résonante. Cet ensemble de valeurs s'appelle le *spectre de Fučik*. Les premiers résultats d'existence de solutions périodiques pour l'équation (10) en situation de résonance, c.-à-d. lorsque (α, β) se trouve sur une des courbes du spectre de Fučik, sont dus à E. N. Dancer [D1] et S. Fučik [Fu]. Ils ont ensuite été généralisés notamment par C. Fabry et A. Fonda [FF]. Il est aisé de vérifier que la limite normalisée des oscillations libres de l'oscillateur asymétrique est donnée par

$$\psi_\beta(t) := \begin{cases} \sin\left(\sqrt{\alpha}\,(t + \frac{\pi}{2\sqrt{\alpha}})\right), & \text{si } t \in \left[-\frac{\pi}{2\sqrt{\alpha}}, \frac{\pi}{2\sqrt{\alpha}}\right], \\ -\frac{\sqrt{\alpha}}{\sqrt{\beta}} \sin\left(\sqrt{\beta}(t - \frac{\pi}{2\sqrt{\alpha}})\right), & \text{si } t \in \left[\frac{\pi}{2\sqrt{\alpha}}, -\frac{\pi}{2\sqrt{\alpha}} + \frac{\pi}{\sqrt{\sigma}}\right]. \end{cases} \tag{11}$$

C. Fabry et A. Fonda [FF] ont démontré que l'existence de solutions 2π-périodiques de l'équation (10) est étroitement liée aux propriétés de la fonction

$$\Phi_\beta(\theta) := \int_0^{2\pi} h(t)\psi_\beta(t + \theta)\, dt.$$

En particulier, lorsque Φ_β ne s'annule pas, l'équation (10) admet au moins une solution 2π-périodique.

Les similitudes entre l'oscillateur singulier (2) et l'oscillateur asymétrique (9) suggèrent plusieurs questions. La première est certainement de savoir ce qu'il advient du Théorème 2 lorsque l'hypothèse (c) est remplacée par l'hypothèse que Φ_1 ne s'annule pas. L'approche utilisée dans [BDC] ne s'étend malheureusement pas au cas $\Phi_1 < 0$. On peut aussi se demander quel rôle joue l'isochronisme ou encore s'il existe une classe d'oscillateurs pour lesquels l'existence de solutions 2π-périodiques de l'équation forcée dépend des propriétés d'une fonction semblable à Φ_1 ou Φ_β.

Une classe plus générale. Christian Fabry a suggéré d'étudier une classe d'oscillateurs qui partagent les propriétés communes des oscillateurs singulier et asymétrique. Dans un travail en collaboration avec lui et D. Smets, nous étudions des perturbations bornées d'oscillateurs isochrones. Il s'agit d'équations du type

$$u'' + V'(u) + g(u) = h(t), \tag{12}$$

où $V :]a, +\infty[\to \mathbb{R}$ est un potentiel strictement convexe $\frac{2\pi}{n}$-isochrone, $g :]a, +\infty[\to \mathbb{R}$ est une fonction bornée, $a < 0$ ou $a = -\infty$ et h est 2π-périodique. Un potentiel V est dit *T-isochrone* si toutes les solutions non constantes de l'équation

$$u'' + V'(u) = 0 \tag{13}$$

sont périodiques de période minimale T. Nous supposons de plus qu'il existe $n \in \mathbb{N}$ tel que, si $a < 0$,

$$\lim_{x \to +\infty} \frac{V'(x)}{x} = \frac{n^2}{4} \quad \text{et} \quad \lim_{x \to a_+} \frac{V'(x)}{x} = +\infty, \tag{14}$$

ou lorsque $a = -\infty$,

$$\begin{cases} \lim\limits_{x \to +\infty} \dfrac{V'(x)}{x} = \alpha > 0 \;\text{ et }\; \lim\limits_{x \to -\infty} \dfrac{V'(x)}{x} = \beta > 0, \\ \text{où } \dfrac{1}{\sqrt{\alpha}} + \dfrac{1}{\sqrt{\beta}} = \dfrac{2}{n}. \end{cases} \tag{15}$$

Lorsque $a \neq -\infty$, le potentiel et la force ont une singularité à droite au point a. Avant de présenter nos résultats, introduisons les fonctions

$$\Gamma(\rho) := \int_0^{2\pi} g(\rho\psi(t))\psi(t)\,dt \tag{16}$$

et

$$\Phi(\theta) := \int_0^{2\pi} h(t)\psi(t+\theta)\,dt\,, \tag{17}$$

où ψ est soit $|\cos(nt/2)|$ si V satisfait (14), soit ψ_β si V satisfait (15). Observons que la fonction Φ est définie à partir de la limite normalisée des oscillations libres de l'oscillateur singulier (2) ou de l'oscillateur asymétrique (9). En fait, pour un oscillateur isochrone satisfaisant l'une des conditions asymptotiques (14) ou (15), la limite normalisée des oscillations libres est indépendante de la forme particulière du potentiel et ne dépend que de son comportement asymptotique en $-\infty$ ou singulier en a et de son comportement asymptotique en $+\infty$.

Existence de solutions périodiques. Dans notre contribution [BFS], nous démontrons le résultat suivant.

Théorème 3. *Supposons que*
(a) $V :]a, +\infty[\to \mathbb{R}$ *est un potentiel strictement convexe* $\frac{2\pi}{n}$*-isochrone vérifiant une des hypothèses* (14) *ou* (15) *pour un certain* $n \in \mathbb{N}$ *et dont la dérivée est localement lipschitzienne ;*
(b) $g :]a, +\infty[\to \mathbb{R}$ *est une fonction bornée, localement lipschitzienne ;*
(c) $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ *est une fonction* 2π*-périodique.*

Notons

$$\Gamma_+ := \liminf_{\rho \to +\infty} \Gamma(\rho) \quad et \quad \Gamma^+ := \limsup_{\rho \to +\infty} \Gamma(\rho), \tag{18}$$

où Γ *est définie par* (16) *et soit* Φ *défini par* (17)*. Alors, nous avons les résultats suivants :*
(i) s'il existe une racine $\Gamma_* \in [\Gamma_+, \Gamma^+]$ *de* Φ *telle que* $\Phi'(\Gamma_*) \neq 0$ *et si le nombre de zéros de* $\Phi - \Gamma_*$ *dans* $[0, \frac{2\pi}{n}[$ *est différent de* 2*, alors l'équation* (12) *possède au moins une solution* 2π*-périodique ;*
(ii) s'il existe deux valeurs régulières $\Gamma_1, \Gamma_2 \in [\Gamma_+, \Gamma^+]$ *de* Φ *telles que les nombres de zéros de* $\Phi - \Gamma_1$ *et* $\Phi - \Gamma_2$ *dans* $[0, \frac{2\pi}{n}[$ *sont différents, l'équation* (12) *possède une suite non bornée de solutions* 2π*-périodiques ;*
(iii) si l'intervalle $[\Gamma_+, \Gamma^+]$ *ne contient aucune valeur critique de* Φ*, l'ensemble des solutions* 2π*-périodiques de* (12) *est borné.*

La preuve de ce théorème est basée sur l'étude d'un système de deux équations du premier ordre équivalent à l'équation (12). L'idée est de remplacer les variables originales (u, u') par des variables du type action-angle en utilisant les oscillations libres associées à l'oscillateur isochrone. Plus précisément, définissons la fonction $\phi(t, \rho)$ comme l'unique solution du problème de Cauchy

$$u'' + V'(u) = 0, \ u(0) = \rho, \ u'(0) = 0.$$

Cette solution est bien entendu $\frac{2\pi}{n}$-périodique. Si u est une solution de l'équation (12) vérifiant pour tout $t \in [0, 2\pi]$, $(V'(u(t)), u'(t)) \neq (0, 0)$, il est possible de définir des fonctions $\rho > 0$ et θ de telle sorte que

$$u(t) = \phi(t + \theta(t), \rho(t)) \quad \text{et} \quad u'(t) = \frac{\partial \phi}{\partial t}(t + \theta(t), \rho(t)) \tag{19}$$

pour tout $t \in [0, 2\pi]$. Une étude précise du système différentiel dans les variables (ρ, θ) et des estimations asymptotiques de la fonction ϕ et de ses dérivées mène, après intégration, aux équations asymptotiques

$$\rho(2\pi) - \rho(0) = \frac{1}{\alpha} \ \Phi'(\theta(0)) + o(1) \qquad \text{pour } \rho(0) \to +\infty\,,$$

$$(\theta(2\pi) - \theta(0))\rho(0) = \frac{1}{\alpha} \ [-\Phi(\theta(0)) + \Gamma(\rho(0))] + o(1) \quad \text{pour } \rho(0) \to +\infty\,,$$

où α est déterminé par l'hypothèse (15) si V satisfait celle-ci, ou bien α vaut $n^2/4$ si V vérifie la condition (14).

Il découle directement de ces estimations asymptotiques que les solutions 2π-périodiques du système en (ρ, θ), et donc aussi les solutions 2π-périodiques de (12), sont a priori bornées lorsque l'intervalle $[\Gamma_+, \Gamma^+]$ ne contient aucune valeur critique de Φ. Si l'intervalle $[\Gamma_+, \Gamma^+]$ contient une valeur régulière Γ_* de Φ alors il existe une suite $\rho_m \to +\infty$ telle que pour m assez grand, les solutions (ρ, θ) telles que $\rho(0) = \rho_m$ et $\theta(0) \in [0, \frac{2\pi}{n}]$ satisfont

$$(\rho(2\pi) - \rho(0), \theta(2\pi) - \theta(0)) \neq (0, 0).$$

Ceci nous permet de faire un calcul de degré dans la région du plan délimitée par l'orbite de l'oscillation libre $\phi(t, \rho_m)$. La fin de la démonstration consiste moralement à calculer le degré de Brouwer de l'application de Poincaré dans les variables (ρ, θ).

La condition d'existence (i) du Théorème 3 généralise certains résultats antérieurs concernant l'oscillateur asymétrique [Fa,FF,FM] et l'oscillateur harmonique [KM].

Certains cas particuliers du Théorème 3 valent la peine d'être mis en évidence. Lorsque g possède une primitive sous-linéaire, il est prouvé dans [KM] que $\Gamma_+ = \Gamma^+ = 0$. Considérons par exemple l'équation singulière

$$u'' - \frac{1}{u^3} + \frac{n^2}{4}u + \sin u = h(t),$$

où $n \in \mathbb{N}$ et l'équation asymétrique

$$u'' + \alpha u^+ - \beta u^- + \sin u = h(t),$$

où α, β satisfont la condition de résonance

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n} \tag{20}$$

pour un certain $n \in \mathbb{N}$. On déduit de la remarque précédente que chacune de ces équations possède au moins une solution 2π-périodique si le nombre de zéros de la fonction Φ dans $[0, \frac{2\pi}{n}[$ est différent de 2, les zéros étant supposés simples.

Un autre cas particulier intéressant est celui d'une perturbation bornée g possédant des limites en $\pm\infty$ (ou uniquement en $+\infty$ sous l'hypothèse (14)). Dans ce cas, un calcul simple donne

$$\Gamma_+ = \Gamma^+ = 2n\sqrt{\alpha}\left(\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\right)$$

lorsque V satisfait (15), et

$$\Gamma_+ = \Gamma^+ = 4g(+\infty)$$

si V satisfait (14). La condition d'existence se formule alors simplement : $\Gamma_+ = \Gamma^+$, donné par les expressions ci-dessus, est une valeur régulière de Φ telle que $\Phi - \Gamma_+$ ne s'annule pas exactement 2 fois dans $[0, \frac{2\pi}{n}[$.

Les résultats énoncés dans le Théorème 3 sont tout à fait explicites lorsque nous considérons une équation forcée

$$u'' + f(u) = h(t), \tag{21}$$

où f est une perturbation bornée d'une force qui dérive d'un potentiel isochrone. Par contre pour une force donnée f, il n'est pas du tout clair si celle-ci rentre dans le cadre de nos résultats. La portée du Théorème 3 serait bien plus importante si nous arrivions à caractériser les forces f qui sont des perturbations bornées de forces dérivant d'un potentiel isochrone. Dans le cas singulier ($a \neq -\infty$) nous avons le résultat suivant. Nous considérons la condition (14) dans le cas $n = 1$ pour fixer les idées, le cas général se traitant de manière similaire.

Théorème 4. *Soit h une fonction 2π-périodique localement intégrable et f une force localement lipschitzienne, singulière au point $-a$, $a > 0$. Notons F la primitive de f qui s'annule en zéro. Supposons de plus que*

(a) $\lim_{x \to -a} F(x) = +\infty$ *;*

(b) *il existe $\tau \in \mathbb{R}$ tel que* $\lim_{x \to +\infty} (f(x) - x/4) = \frac{\tau}{4}$ *;*

(c) *il existe $\delta > 0$ tel que pour tout $x \in (-a, -a + \delta)$:*

$$f'(x) > 0 \quad et \quad |f'(x)| < F(x)^{-3/2} |f(x)|^3 / \sqrt{2}.$$

Alors, si les zéros de $\Phi - (\tau - a)$ sont simples, où Φ est défini par (17), *et leur nombre dans $[0, 2\pi[$ est différent de 2, l'équation* (21) *possède au moins une solution 2π-périodique.*

La démonstration de ce théorème utilise une caractérisation des potentiels isochrones que l'on peut trouver notamment dans un travail récent de S. Bolotin et R. MacKay [BM] selon laquelle un potentiel strictement convexe V est T-isochrone si et seulement si le réarrangement symétrique de son graphe est celui du potentiel harmonique $2\pi^2 u^2 / T^2$. Pour prouver le Théorème 4, il suffit dès lors de montrer que la force f peut s'écrire $V'(u) + g(u)$, où V est un potentiel strictement convexe tel que $V(0) = V'(0) = 0$, pour tout $y > 0$

$$V^{-1}(y) = \{u_1, u_2\}, \text{ où } u_1 < 0 < u_2 \text{ sont tels que } u_2 - u_1 = 2\sqrt{2y},$$

et

$$\lim_{u \to +\infty} g(u) = \frac{\tau - a}{4}.$$

Les hypothèses énoncées dans la condition (c) permettent de construire un potentiel strictement convexe 2π-isochrone qui coïncide avec F sur un voisinage à droite de $-a$. Grâce à la première, F est strictement convexe proche de $-a$ tandis que la seconde assure que la branche isochrone correspondante pour $u \to +\infty$ est un morceau de graphe convexe.

En translatant la singularité à l'origine, le Théorème 4 permet de traiter l'équation singulière modèle

$$u'' - \frac{1}{u^\nu} + \frac{1}{4}u = h(t) \tag{22}$$

pour $\nu \geq 1$. On obtient donc une amélioration partielle du Théorème 2 pour cette équation modèle : si h est telle que la fonction Φ_1 ne s'annule pas exactement 2 fois dans $[0, 2\pi[$, les zéros étant supposés simples, alors l'équation (22) possède au moins une solution 2π-périodique.

Les conditions suffisantes d'existence établies dans le Théorème 3 ne sont pas optimales comme nous le verrons plus loin dans l'exposé. Cependant, le Théorème 3 est faux si ces conditions ne sont pas remplies. En effet, nous avons prouvé dans [BFS] que l'équation

$$u'' + V'(u) = \varepsilon \sin t \tag{23}$$

ne possède aucune solution 2π-périodique si ε est suffisamment petit (et différent de zéro) et V est un potentiel strictement convexe, deux fois différentiable en son point critique, 2π-isochrone et satisfaisant (14) ou (15) avec $n = 1$. En particulier, l'équation

$$u'' - \frac{1}{u^3} + \frac{1}{4}u = \varepsilon \sin t$$

n'a pas de solution 2π-périodique pour $\varepsilon \neq 0$ suffisamment petit.

Solutions périodiques de grande amplitude. Outre l'existence d'une famille non bornée d'oscillations libres et la non-existence de solution périodique de période 2π pour certaines forces extérieures, d'autres caractéristiques des oscillateurs en situation de résonance sont importantes, surtout aux yeux des physiciens. En effet, une question essentielle est celle de l'existence de solutions périodiques de grande amplitude en présence d'une force additionnelle de friction. Considérons par exemple l'équation amortie

$$u'' + \varepsilon u' + u = \sin t.$$

Lorsque $\varepsilon \neq 0$, l'équation possède une solution 2π-périodique. Cette solution disparaît bien évidemment lorsque $\varepsilon \to 0$ et on vérifie sans peine que l'amplitude de la solution périodique de l'équation amortie est de l'ordre de $1/\varepsilon$ pour $\varepsilon \sim 0$.

Dans notre travail [BFS], nous traitons l'équation amortie

$$u'' + \varepsilon f(u)u' + V'(u) + g(u) = h(t) \tag{24}$$

dans une situation de résonance comme précédemment, et nous cherchons des conditions précises sous lesquelles apparaissent des solutions

2π-périodiques d'amplitude d'ordre $1/\varepsilon$ pour $\varepsilon \to 0$. En utilisant des arguments similaires à ceux invoqués pour démontrer le Théorème 3, nous obtenons le résultat suivant.

Théorème 5. *Supposons que*
(a) $V :]a, +\infty[\to \mathbb{R}$ *est un potentiel strictement convexe,* $\frac{2\pi}{n}$*-isochrone, vérifiant une des hypothèses* (14) *ou* (15) *pour un certain* $n \in \mathbb{N}$ *et dont la dérivée est localement lipschitzienne ;*
(b) $g :]a, +\infty[\to \mathbb{R}$ *est une fonction bornée, localement lipschitzienne ;*
(c) $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ *est une fonction* 2π*-périodique ;*
(d) $f :]a, +\infty[\to \mathbb{R}$ *est une fonction bornée, continue, telle qu'il existe* $R, \eta > 0$ *vérifiant* $f(u) \geq \eta$ *ou* $f(u) \leq -\eta$ *pour* $|u| \geq R$ *dans le cadre de l'hypothèse* (15) *ou l'une des inégalités pour* $u \geq R$ *uniquement, si* V *satisfait* (14).
Soient Γ *défini par* (16), Φ *défini par* (17) *et* Γ_+, Γ^+ *définis par* (18). *Alors, nous avons les résultats suivants :*
(i) s'il existe une valeur régulière $\Gamma_* \in]\Gamma_+, \Gamma^+[$ *de* Φ *telle que* $\Phi - \Gamma_*$ *s'annule en au moins un point de* $[0, \frac{2\pi}{n}[$*, alors il existe* $K > 0$ *et* $\varepsilon_0 > 0$*, dépendant de* n*, tels que si* $|\varepsilon| \leq \varepsilon_0$*, l'équation* (24) *a une solution* 2π*-périodique qui peut s'écrire sous la forme* (19)*, avec*

$$\rho(t) \geq \frac{K}{|\varepsilon|}$$

pour tout $t \in [0, 2\pi]$ *;*
(ii) si $[\Gamma_+, \Gamma^+] \cap [\min \Phi, \max \Phi] = \emptyset$*, alors il existe* $L > 0$ *et* $\varepsilon_1 > 0$*, dépendant de* n*, tels que si* $|\varepsilon| \leq \varepsilon_1$*, toute solution* 2π*-périodique* u *de l'équation* (24) *satisfait*

$$\|u\|_\infty \leq L.$$

Bornitude des solutions. Une autre question importante est celle de l'existence de solutions non bornées sans la présence de frottement. La bornitude de toutes les solutions d'une équations du type

$$u'' + W(u) = h(t),$$

où h est une force extérieure périodique est un problème classique connu sous le nom de *Problème de Littlewood.* C'est J. E. Littlewood [Li1,Li2] qui a le premier posé cette question, ouvrant la voie à de nombreux travaux. G. R. Morris [Mo] a tout d'abord démontré que toutes les solutions sont bornées lorsque $W(u) = u^3$ et h est continue. M. Levi [Le] a ensuite atteint la même conclusion pour une classe assez large de fonctions W. Plus récemment, B. Liu [Liu1,Liu2] a étudié l'oscillateur asymétrique et le cas d'une nonlinéarité W sous-linéaire.

Intéressons-nous momentanément à l'oscillateur harmonique. Il est bien connu que lorsque le terme forçant est de même fréquence que les oscillations libres, toutes les solutions sont non bornées. Dans le cas contraire, toutes les solutions sont bornées. Dans les problèmes résonants non linéaires, la situation n'est pas aussi simple. En effet, il peut y avoir coexistence de solutions bornées et non bornées pour une même force extérieure [AO], [FM]. D'autre part un résultat de Massera [Ma] permet de conclure que pour une grande classe d'équations forcées, la non-existence de solutions 2π-périodiques implique que toutes les solutions sont non bornées. Ce résultat s'applique en particulier à l'équation (23).

Dans deux autres travaux [BF1], [BF2] en collaboration avec C. Fabry, nous étudions le caractère borné des solutions des oscillateurs isochrones en situation de résonance. Dans [BF1], nous montrons que les solutions de grande amplitude de (12) sont non bornées dans la situation qui suit.

Théorème 6. *Supposons que*

(a) $V :]a, +\infty[\to \mathbb{R}$ *est un potentiel strictement convexe* $\frac{2\pi}{n}$*-isochrone vérifiant une des hypothèses* (14) *ou* (15) *pour un certain* $n \in \mathbb{N}$ *et dont la dérivée est localement lipschitzienne ;*

(b) $g :]a, +\infty[\to \mathbb{R}$ *est une fonction bornée, localement lipschitzienne ;*

(c) $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ *est une fonction* 2π*-périodique.*

Soient Γ *défini par* (16), Φ *défini par* (17) *et* Γ_+, Γ^+ *définis par* (18). *Supposons que* $\max \Phi > \Gamma^+$ *et* $\min \Phi < \Gamma_+$. *Si l'intervalle* $[\Gamma_+, \Gamma^+]$ *ne contient pas de valeurs critiques de* Φ, *alors il existe* $R > 0$ *tel que toute solution* u *de* (12) *satisfaisant* $(u(0))^2 + (u'(0))^2 > R$ *est non bornée, soit dans le futur soit dans le passé.*

En particulier, lorsque $\Gamma_+ = \Gamma^+ = 0$, les solutions de grande amplitude sont non bornées dès que Φ s'annule (les zéros étant supposés simples). Lorsque le nombre de zéros de Φ est différent de deux, nous sommes donc dans une situation de coexistence de solutions bornées et non bornées. En effet, d'après le Théorème 3, il existe au moins une solution 2π-périodique. La preuve du Théorème 6 se base sur une étude du système d'équations du premier ordre dans les variables action-angle (ρ, θ).

Lorsque Φ est de signe constant, certains arguments laissent à penser que toutes les solutions sont bornées bien que peu de résultats précis soient établis. Lorsque la force extérieure h est régulière (au moins de classe C^4), B. Liu [Liu1] a démontré que toutes les solutions de l'oscillateur asymétrique forcé (10) sont bornées. Ce résultat est également valable pour un oscillateur isochrone forcé

$$u'' + V'(u) = h(t) \tag{25}$$

pourvu que le potentiel soit suffisamment régulier. En effet, le théorème suivant est prouvé dans [BF2]. La démonstration est basée sur le *Small Twist theorem* de R. Ortega [O1], [O2] qui améliore certains résultats de la célèbre *théorie KAM*[1] attribués à A. N. Kolmogorov, V. I. Arnold et J. K. Möser.

THÉORÈME 7. *Soit* $V \in C^6(\mathbb{R})$ *un potentiel* 2π*-isochrone strictement convexe et* $h \in C^6(\mathbb{R})$ *une fonction* 2π*-périodique. Supposons que* V *satisfasse*

$$\lim_{x\to+\infty} V''(x) = \alpha > 0, \lim_{x\to-\infty} V''(x) = \beta > 0$$

et

$$\lim_{|x|\to+\infty} x^4 V^{(6)}(x) = 0,$$

où le couple $(\alpha, \beta) \in \mathbb{R}^+$ *vérifie la condition de résonance* (20) *pour un certain entier* $n \geq 1$*. Si la fonction* Φ*, définie par* (17)*, ne change pas de signe, alors toutes les solutions de* (25) *sont bornées, c.-à-d. que si* u *est une solution de* (25)*, elle est définie sur tout* $\mathbb{R}$ *et*

$$\sup_{t\in\mathbb{R}}(|u(t)| + |u'(t)|) < +\infty.$$

En fait, lorsque Φ est de signe constant, nous sommes dans une situation de type KAM [O1], [O2]. Il est en outre possible de démontrer dans certains cas particuliers l'existence de solutions quasi-périodiques de grande amplitude et d'une suite de solutions de période minimale multiple de 2π, appelées solutions sous-harmoniques, dont la période et l'amplitude tendent vers $+\infty$ [FM,BF1]. Ces solutions bornées de grandes amplitudes jouent le rôle de "barrières" emprisonnant les autres solutions dans le plan de phase.

L'oscillateur avec obstacle

Nous avons mentionné précédemment que l'oscillation libre normalisée de l'oscillateur singulier converge, lorsque l'amplitude tend vers l'infini, vers une solution d'un problème avec obstacle. En effet, cette fonction limite $\psi_b(t) = |\cos\sqrt{b}t|$ satisfait l'équation linéaire

$$u'' + bu = 0 \tag{26}$$

tant qu'elle ne s'annule pas. Si la fonction ψ_b décrit la trajectoire d'un point matériel, un zéro peut être assimilé à un impact du point matériel sur l'obstacle $u = 0$. Observons que les zéros de la fonction ψ_b décrivent des rebonds parfaitement élastiques. En effet, la vitesse avant l'impact $\psi_b'(t_0^-)$ est égale en module à la vitesse après l'impact : $\psi_b'(t_0^+) = -\psi_b'(t_0^-)$.

La fonction ψ_b est donc une oscillation libre de l'oscillateur (26) soumis aux conditions non linéaires

$$\begin{aligned} u(t) \geq 0 \text{ pour tout } t \in \mathbb{R}, \\ u(t_0) = 0 \Rightarrow u'(t_0^-) = -u'(t_0^+). \end{aligned} \tag{27}$$

[1] voir [dlL] pour une introduction

La dernière condition traduit physiquement une collision élastique sur l'obstacle, c.-à-d. sans perte d'énergie.

Considérons à présent un oscillateur forcé

$$u'' + bu = h(t) \tag{28}$$

et le mouvement d'un point matériel satisfaisant l'équation (28), restreinte au demi-axe $u \geq 0$, et soumis aux conditions (27) lors d'un impact sur l'obstacle $u = 0$. La trajectoire de la particule satisfait l'équation (28) uniquement lorsqu'elle ne touche pas l'obstacle. Dès lors, l'ensemble des conditions (27) est compatible avec la trajectoire d'une particule au repos sur l'obstacle. Physiquement cela correspond à une particule maintenue contre l'obstacle par une force dirigée vers le demi-axe $u < 0$. Par conséquent, une solution du problème qui s'annule sur un intervalle n'a de sens que si sur cet intervalle la force extérieure h est négative. Ceci nous conduit à une définition plus précise de *solution* du problème.

Définition 3. Soient $b \in \mathbb{R}^+$ et $g : [0, 2\pi] \times \mathbb{R}^+ \to \mathbb{R}$ une fonction de Carathéodory telle que pour une certaine fonction $p \in L^2(0, 2\pi)$,

$$|g(t, u)| \leq p(t)$$

pour tout $u \in \mathbb{R}_0^+$ et pour presque tout $t \in [0, 2\pi]$.

La fonction $u \in C([0, 2\pi])$ est une *solution admissible* du problème (29) avec obstacle si

(i) u satisfait l'équation

$$u'' + bu = g(t, u) \tag{29}$$

pour presque tout $t \in]0, 2\pi[$ tel que $u(t) > 0$;

(ii) si t_0 est un zéro isolé de u, alors la dérivée à gauche $u'(t_0^-)$ et la dérivée à droite $u'(t_0^+)$ existent et $u'(t_0^-) = -u'(t_0^+)$;

(iii) si t_0 est un zéro de u, isolé à gauche (respectivement à droite), $u'(t_0^-)$ (respectivement $u'(t_0^+)$) existe, et si $u'(t_0^-)$ (respectivement $u'(t_0^+)$) est différent de 0, t_0 est un zéro isolé de u (et $u'(t_0^-) = -u'(t_0^+)$);

(iv) si $u(t) = 0$ pour tout t dans un intervalle I,

$$\int_J g(t, 0)\, dt \leq 0,$$

sur tout sous-intervalle J de I.

Conformément à cette définition, l'équation (29) admet la solution triviale si et seulement si $g(t, 0) \leq 0$ pour presque tout $t \in [0, 2\pi]$.

L'oscillateur (28) soumis à un obstacle à l'origine a fait l'objet de divers travaux de recherche. A. C. Lazer et P. J. McKenna [LMc] se sont intéressés à l'existence de mouvements périodiques en présence d'une force de frottement. R. Ortega [O3] traite le problème de la bornitude des solutions. H. Lamba [La] présente un panorama général de la dynamique d'un oscillateur avec obstacle soumis à des forces extérieures harmoniques. En collaboration avec C. Fabry, nous étudions dans [BF3] les similitudes avec l'oscillateur asymétrique et présentons des résultats d'existence de solutions périodiques.

Existence de solutions périodiques. Il suit des discussions précédentes que le problème 2π-périodique associé à l'oscillateur (29) avec obstacle en zéro est résonant pour les mêmes valeurs de b que dans le cas de l'oscillateur singulier, c.-à-d. pour $b \in \Sigma$. D'autre part, rappelons que la condition de résonance associée à la période 2π pour le potentiel quadratique asymétrique

$$V(u) = \alpha\frac{(u^+)^2}{2} + \beta\frac{(u^-)^2}{2},$$

où $\alpha > 0$ et $\beta > 0$, est donnée par

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}, \quad n \in \mathbb{N}.$$

Il est clair que chaque courbe $C_n \subset \mathbb{R}^2$ déterminée par cette dernière équation est asymptotique aux droites d'équation $\alpha = \frac{n^2}{4}$ et $\beta = \frac{n^2}{4}$. Par ailleurs, lorsque par exemple $\beta \to +\infty$, la force de rappel βu^- tend ponctuellement vers l'infini sauf à l'origine, et grossièrement, un "mur" (l'obstacle) apparaît en zéro dans le potentiel limite. Autrement dit, l'oscillateur avec obstacle peut être considéré comme la limite d'un oscillateur asymétrique dans lequel le coefficient de la force de rappel à gauche de l'origine tend vers l'infini. C'est dans cet esprit que nous traitons le problème dans [BF3]. Cette approche avait déjà été suggérée par A. C. Lazer et P. J. McKenna mais curieusement non exploitée.

Lorsque $0 < b \neq \frac{n^2}{4}$ pour tout $n \in \mathbb{N}_0$, l'existence d'une solution admissible de l'équation (29) avec obstacle à l'origine s'obtient aisément comme limite, au sens $H^1(0, 2\pi)$, de solutions de la suite d'équations

$$u'' + \alpha_m u^+ - \beta_m u^- = \tilde{g}(t, u), \tag{30}$$

où les coefficients α_m, β_m satisfont la relation

$$\frac{1}{\sqrt{\alpha_m}} + \frac{1}{\sqrt{\beta_m}} = \frac{1}{\sqrt{b}},$$

$\beta_m \to +\infty$ et $\tilde{g}$ est une extension Carathéodory de g sur $[0, 2\pi] \times \mathbb{R}$ majorée par une fonction de $L^2(0, 2\pi)$. Le seul point quelque peu délicat consiste à montrer que les éventuels rebonds de la fonction limite sur l'obstacle sont élastiques. Ceci se déduit d'un argument d'énergie, la force extérieure $\tilde{g}$ étant bornée par une fonction L^2. Nous pouvons donc énoncer le théorème suivant dont la preuve complète est présentée dans [BF3].

THÉORÈME 8. *Soient $b \in \mathbb{R}_0^+$ et $g : [0, 2\pi] \times \mathbb{R}^+ \to \mathbb{R}$ une fonction de Carathéodory telle que pour une certaine fonction $p \in L^2(0, 2\pi)$,*

$$|g(t, u)| \leq p(t)$$

pour tout $u \in \mathbb{R}^+$ et pour presque tout $t \in [0, 2\pi]$. Si $b \neq \frac{n^2}{4}$ pour tout $n \in \mathbb{N}_0$, le problème (29) *avec obstacle en zéro possède une solution admissible, au sens de la Définition* 3, *2π-périodique.*

Considérons à présent les cas résonants ($b = \frac{n^2}{4}$, $n \in \mathbb{N}_0$). Étant donné les observations précédentes et les similitudes avec les oscillateurs singulier et asymétrique, il semble naturel que les conditions de non-résonance pour le problème avec obstacle s'obtiennent en passant à la limite pour $\beta \to +\infty$ dans celles relatives à l'oscillateur asymétrique. Par ailleurs, la limite normalisée des oscillations libres de l'oscillateur avec obstacle est, par homogénéité, l'oscillation de norme 1, c.-à-d. la limite normalisée des oscillations libres de l'oscillateur singulier. Ces deux points de vue nous mènent à considérer, pour l'oscillateur forcé (28) avec $b = \frac{n^2}{4}$, la même fonction

$$\Phi_{n,h}(\theta) := \int_0^{2\pi} h(t) \left| \cos \frac{n}{2}(t + \theta) \right| dt. \tag{31}$$

Si nous voulons traiter l'équation plus générale (29), nous devons tenir compte du fait que la force extérieure $g(t, u)$ dépend elle-même de la position. Dans ce cas, nous définissons les fonctions γ_+, γ^+ pour presque tout $t \in [0, 2\pi]$, par

$$\gamma_+(t) := \liminf_{x \to +\infty} g(t, x) \quad \text{ et } \quad \gamma^+(t) := \limsup_{x \to +\infty} g(t, x),$$

et nous considérons la famille de fonctions (31), où h est une fonction mesurable quelconque satisfaisant

$$\gamma_+(t) \leq h(t) \leq \gamma^+(t), \text{ pour presque tout } t \in [0, 2\pi]. \tag{32}$$

Remarquons que si les zéros de $\Phi_{n,h}$ sont simples quel que soit h satisfaisant (32), le nombre de zéros ne dépend pas du choix particulier de h. Les conditions de non résonance s'énoncent de la façon suivante.

THÉORÈME 9. *Soient $b = \frac{n^2}{4}$, $n \in \mathbb{N}_0$ et $g : [0, 2\pi] \times \mathbb{R}^+ \to \mathbb{R}$ une fonction de Carathéodory telle que pour une fonction $p \in L^2(0, 2\pi)$,*

$$|g(t, u)| \leq p(t)$$

pour tout $u \in \mathbb{R}^+$ et pour presque tout $t \in [0, 2\pi]$. Si pour toute fonction mesurable h vérifiant les inégalités (32), *la fonction $\Phi_{n,h}$ définie par* (31) *ne possède que des zéros simples et que ce nombre de zéros dans $[0, \frac{2\pi}{n}[$ est différent de 2, alors il existe une solution admissible 2π-périodique de l'équation* (29) *avec obstacle en zéro.*

Comme dans le cas non résonant, la preuve se base sur un processus d'approximation à partir de solutions des équations (30). Lorsque $\tilde{g}$ ne dépend pas de u, l'existence de solutions périodiques approximantes se déduit du Théorème 3. En effet, si le nombre de zéros de $\Phi_{n,h}$ dans $[0, \frac{2\pi}{n}[$ est différent de 2, il en est de même pour le nombre de zéros de la fonction $\Phi_{\beta_m,h}$ définie par

$$\Phi_{\beta_m,h}(\theta) := \int_0^{2\pi} h(t)\psi_{\beta_m}(t+\theta)\,dt,$$

pourvu que β_m soit suffisamment grand. Pour rappel, la fonction ψ_{β_m} est l'oscillation libre associée à l'équation (30) :

$$\psi_{\beta_m}(t) := \begin{cases} \sin\left(\sqrt{\alpha_m}\,(t+\frac{\pi}{2\sqrt{\alpha_m}})\right), & \text{si } t \in \left[-\frac{\pi}{2\sqrt{\alpha_m}}, \frac{\pi}{2\sqrt{\alpha_m}}\right], \\ -\frac{\sqrt{\alpha_m}}{\sqrt{\beta_m}}\sin\left(\sqrt{\beta_m}(t-\frac{\pi}{2\sqrt{\alpha_m}})\right), & \text{si } t \in \left[\frac{\pi}{2\sqrt{\alpha_m}}, -\frac{\pi}{2\sqrt{\alpha_m}}+\frac{\pi}{\sqrt{\sigma}}\right]. \end{cases}$$

En fait, il est facile de vérifier que

$$\Phi_{\beta_m,h} \longrightarrow \Phi_{n,h} \ \text{ dans } C^1([0, \frac{2\pi}{n}]), \text{ lorsque } \beta_m \to +\infty. \tag{33}$$

Lorsque $\tilde{g}$ dépend de u, les résultats d'existence nécessaires pour s'assurer que les équations (30) possèdent au moins une solution 2π-périodique ont été démontrés par C. Fabry et A. Fonda [FF]. La condition d'existence demande que les zéros de la fonction $\Phi_{\beta_m,h}$ soient simples et que leur nombre dans $[0, \frac{2\pi}{n}[$ soit différent de 2 quel que soit h vérifiant les inégalités (32). Cette dernière condition est satisfaite lorsque les hypothèses du Théorème 9 sont remplies. En effet, la convergence (33) est dans ce cas uniforme dans l'ensemble des fonctions mesurables h vérifiant les inégalités (32).

Comme dans les problèmes non linéaires résonants dont nous avons discuté précédemment, les conditions de non résonance décrites dans le Théorème 9 sont optimales, bien qu'elles ne soient probablement pas nécessaires. Effectivement, lorsque pour une certaine force extérieure h, la fonction $\Phi_{n,h}$ possède exactement deux zéros dans $[0, \frac{2\pi}{n}[$, supposés simples, l'existence d'une solution admissible 2π-périodique n'est pas assurée comme le montre l'exemple suivant.

THÉORÈME 10. *Le problème avec obstacle en zéro associé à l'équation*

$$u'' + \frac{u}{4} = \sin t,$$

ne possède pas de solution admissible 2π*-périodique.*

La démonstration de ce théorème, qui suit essentiellement un raisonnement par l'absurde, peut être consultée dans [BF3].

Solutions périodiques à rebonds multiples. La définition de solution admissible ne permet pas de déduire du Théorème 8 et du Théorème 9 l'existence d'une solution non triviale lorsque $g(t,0)$ est une fonction négative. Par exemple, le problème

$$u'' + bu = -1$$

avec obstacle en zéro possède la solution $u(t) = 0$ correspondant aux conditions initiales $u(t_0) = u'(t_0) = 0$. Cette solution est la limite des équilibres $u_m = -1/\beta_m$, solutions de la suite d'équations

$$u'' + \alpha_m u^+ - \beta_m u^- = -1.$$

Un calcul explicite permet pourtant d'obtenir d'autres solutions non triviales

$$u(t) = \frac{-1}{b} + \frac{\cos\left(\sqrt{b}(t - \pi/r)\right)}{b\cos\left(\sqrt{b}\pi/r\right)},$$

étendues par $\frac{2\pi}{r}$-périodicité avec $r \in \mathbb{N}$ et $r > 2\sqrt{b}$. Ces solutions ont exactement r impacts dans l'intervalle $[0, 2\pi[$.

Nous complétons à présent le Théorème 8 et le Théorème 9 par un résultat d'existence de solutions de ce type pour des termes forçants $h(t)$ négatifs. En général, nous pouvons obtenir pour tout entier $r > 2\sqrt{b}$ une solution qui possède r impacts dans $[0, 2\pi[$.

THÉORÈME 11. *Soient $b > 0$ et h une fonction 2π-périodique, continue et négative. Pour tout entier $r > 2\sqrt{b}$, le problème avec obstacle en zéro associé à l'équation*

$$u'' + bu = h(t)$$

a au moins une solution 2π-périodique admissible u ayant r impacts dans $[0, 2\pi[$. *De plus, chaque rebond est non tangent à l'obstacle, c.-à-d.*

$$u'(t_i^-) = -u'(t_i^+) \neq 0$$

pour chaque zéro t_i, $i = 1, \ldots, r$.

La stratégie que nous adoptons pour obtenir ces solutions se base à nouveau sur le schéma approximatif

$$u'' + \alpha_m u^+ - \beta_m u^- = h(t). \tag{34}$$

Néanmoins, nous avons cette fois besoin d'une information nodale sur les solutions des équations (34) pour s'assurer que la fonction limite possède bien r zéros. Pour obtenir les solutions approximantes, nous utilisons une version du Théorème de Poincaré-Birkhoff due à C. Rebelo et F. Zanolin [RZ]. Pour expliquer son utilisation, nous avons besoin d'introduire la notation suivante. Si $z : [0, 2\pi] \to \mathbb{R}^2$ est une fonction continue telle que $z(t) \neq 0$ pour tout $t \in [0, 2\pi]$, nous notons $\mathrm{rot}(z)$ le nombre de tours que fait la courbe $z(t)$ autour de l'origine dans le sens horlogique. Les solutions approximantes que nous cherchons sont donc des fonctions u_m solutions de (34), avec $\beta_m \to +\infty$, telles que $\mathrm{rot}(u_m) = r$. Lorsque β_m est assez grand, les solutions de (34) de petite énergie initiale (voir la définition précise dans [BF3]) font un grand nombre de tours autour de l'origine dans le plan de phase tandis que les solutions de grande énergie initiale font peu de tours. Des estimations précises, uniformes pour β_m assez grand, permettent en réalité de montrer que pour tout entier q, il existe E_q tel que toute solution u de (34) d'énergie initiale E_q, vue comme vecteur $(u, u') \in \mathbb{R}^2$, satisfait $\mathrm{rot}(u) > q$. D'autre part, pour tout $s > 0$, il existe E_s^* tel que toute solution d'énergie initiale E_s^* satisfait $\mathrm{rot}(u) < (2+s)\sqrt{b}$. Ici le nombre de tours est interprété dans un sens généralisé comme la variation angulaire divisée par 2π (donc non nécessairement entier). C'est ici qu'intervient le Théorème de Poincaré-Birkhoff qui est un résultat du type “valeur intermédiaire”. En effet, il permet d'affirmer l'existence d'une solution 2π-périodique u telle que $\mathrm{rot}(u) = r$ dès qu'il existe dans le plan de phase deux courbes fermées, de conditions initiales menant l'une à des solutions faisant un nombre de tours autour de l'origine plus grand que r, l'autre à des solutions dont le nombre de tours est plus petit que r. Les estimations évoquées ci-dessus suggèrent bien évidemment de choisir des courbes d'énergie constante $E = E_r$ et $E = E_s^*$ avec $(2+s)\sqrt{b} < r$. Nous pouvons dès lors conclure qu'il existe une solution 2π-périodique tournant r fois autour de l'origine quel que soit $r > 2\sqrt{b}$.

Définissons à présent une suite u_m de solutions 2π-périodiques de la suite d'équations (34) pour $\beta_m \to +\infty$ de telle sorte que chaque u_m soit positif sur r intervalles disjoints. Un contrôle uniforme sur l'énergie des solutions u_m permet de s'assurer que la longueur de ces intervalles ne tend pas vers zéro si bien que la fonction limite a le nombre de zéros souhaité. Par ailleurs, le contrôle sur l'énergie implique aussi que la valeur de $|u_m'|$ est strictement positive, uniformément en m, là où u_m s'annule. Nous en déduisons que les rebonds de la fonction limite sont non tangents.

Les autres aspects des problèmes résonants isochrones que nous avons abordés concernant les oscillateurs asymptotiquement asymétrique ou singulier se retrouvent également dans le problème avec obstacle. La question relative à la bornitude des solutions a été traitée par R. Ortega [O3]. Les résultats sont similaires à ceux que nous avons énoncés dans le cadre des oscillateurs singulier et asymétrique. Par contre, le problème en présence d'un terme de friction n'a été que peu étudié [LMc].

Equations contenant une nonlinéarité périodique

Considérons le problème 2π-périodique associé à l'équation

$$u'' + u + \sin u = h(t) + a\sin(t - \phi), \tag{35}$$

où $a \in \mathbb{R}$ et h est une fonction orthogonale au noyau de l'opérateur

$$L : u \to u'' + u,$$

c.-à-d. telle que

$$\int_0^{2\pi} h(t)\sin t\,dt = \int_0^{2\pi} h(t)\cos t\,dt = 0.$$

Nous avons vu que pour $g(u) = \sin u$, les limites Γ_+ et Γ^+ sont nulles. Par ailleurs, puisque h est orthogonal au noyau, nous pouvons calculer aisément la fonction $\Phi(\theta) = -\pi\sin(\theta + \phi)$. Nous avons donc affaire à une équation pour laquelle le Théorème 3 est impuissant.

Cependant le fait que la nonlinéarité $\sin u$ soit périodique a été exploité par E. N. Dancer [D2] et J. R. Ward [W] pour traiter le problème de Dirichlet

$$u'' + u + \sin u = h(t) + a\sin t,$$
$$u(0) = u(\pi) = 0,$$

où, cette fois, la fonction h est orthogonale à la fonction propre $\sin t$:

$$\int_0^{\pi} h(t)\sin t\,dt = 0.$$

Lorsque a est suffisamment petit, le problème possède une solution. Ce problème de Dirichlet, bien que résonant, est d'un certain point de vue fort différent du problème périodique. En effet, le noyau de l'opérateur différentiel est unidimensionnel alors qu'il est de dimension deux dans le cas périodique. Cette différence rend les arguments de E. N Dancer et J. R. Ward inapplicables. Un résultat abstrait de D. Lupo et S. Solimini [LuS] s'applique au problème périodique mais uniquement au cas $a = 0$.

Résultats abstraits. Gardant comme motivation principale la résolution du problème périodique (35) pour a petit, nous nous sommes concentrés dans un travail en collaboration avec C. Fabry et D. Ruiz [BFR] sur le problème abstrait suivant. Considérons un ouvert $\Omega \subset \mathbb{R}^N$ et $(X, \|\cdot\|)$ un espace de Hilbert de fonctions réelles de carré sommable définies dans Ω. L'espace X pourrait être

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\}$$

muni de sa norme usuelle

$$\|u\|_{H^1(\Omega)} := \sqrt{\int_\Omega (|\nabla u|^2 + u^2)\,dx}$$

ou

$$H_0^1(\Omega) := \mathrm{adh}_{\|\cdot\|_{H^1(\Omega)}} C_c^\infty(\Omega)$$

muni de sa norme usuelle

$$\|u\|_{H_0^1(\Omega)} := \sqrt{\int_\Omega |\nabla u|^2 \, dx}$$

ou encore

$$H_{2\pi}^1(0, 2\pi) := \{u \in H^1(0, 2\pi) \mid u(0) = u(2\pi)\}$$

muni de la norme de $H^1(0, 2\pi)$. Nous utilisons la notation $Y^\perp$ pour désigner le complémentaire orthogonal d'un sous-ensemble $Y \subset X$, l'orthogonalité étant entendue au sens du produit scalaire usuel de $L^2(\Omega)$. Nous avons donc

$$Y^\perp = \{f \in X | \int_\Omega f \, y \, dx = 0 \text{ pour tout } y \in Y\}.$$

Considérons aussi un opérateur linéaire auto-adjoint tel que pour un certain $\rho > 0$ l'opérateur $L[u] + \rho u$ soit elliptique, c.-à-d. tel que

$$\int_\Omega \left(L[u]\, u + \rho u^2\right) \, dx \geq c\|u\|^2$$

pour un certain $c > 0$. Il est bien connu que sous cette condition d'ellipticité, l'opérateur L possède une suite croissante de valeurs propres $(\mu_n)_n \in \mathbb{R}$ de multiplicité finie. Notons X_n les sous-espaces propres correspondants. Nous nous intéressons à l'équation

$$L[u] - \mu_n u + g(u) = h + w, \tag{36}$$

où g est une fonction périodique continue de valeur moyenne nulle, $h \in X_n^\perp$ et $w \in X_n$. L'équation (36) est l'équation d'Euler-Lagrange associée à la fonctionnelle $I_w : X \to \mathbb{R}$ définie par

$$I_w(u) := \int_\Omega \left(\frac{1}{2}(L[u]\, u - \mu_n u^2) + G(u) - (h + w)u\right) \, dx, \tag{37}$$

où G est la primitive de g de valeur moyenne nulle. Afin d'utiliser une généralisation du Théorème de Riemann-Lebesgue due à S. Solimini [S], nous supposons que les fonctions propres satisfont l'hypothèse suivante :

$$\text{si } w \in X_n, \; w \in C^1(\Omega) \text{ et } |\nabla w(x)| \neq 0 \text{ pour presque tout } x \in \Omega. \tag{38}$$

Cette hypothèse exclut par exemple les fonctions propres constantes. Dès lors, lorsque nous considérons des conditions aux limites périodiques et l'opérateur $u \to u''$, notre approche ne nous permet pas de considérer le problème résonant à la première valeur propre ($\mu = 0$). Pour toutes les autres valeurs propres, les fonctions propres correspondantes vérifient l'hypothèse précitée.

Avant de décrire la géométrie de la fonctionnelle I_w, intéressons-nous aux propriétés de compacité des suites de Palais-Smale, c.-à-d. des suites

$(u_n)_n \subset X$ telles que $I_w(u_n) \to c$ pour une certaine valeur c, que nous appellerons *niveau*, et $I'_w(u_n) \to 0$. En utilisant la version de S. Solimini du Théorème de Riemann-Lebesgue, nous démontrons que les suites de Palais-Smale possèdent une sous-suite convergente sauf si $w = 0$ et c est le niveau de l'unique point critique $\bar{u}$ de la fonctionnelle

$$\bar{I}(u) := \int_\Omega \left(\frac{1}{2}(L[u]\, u - \mu_n u^2) - hu \right) dx$$

dans $X_n^\perp$. Nous pouvons à présent énoncer notre résultat d'existence de points critiques. Nous décomposons l'espace X en somme directe

$$X = X_- \oplus X_n \oplus X_+,$$

où

$$X_- := X_1 \oplus \cdots \oplus X_{n-1} \quad \text{et} \quad X_+ := (X_- \oplus X_n)^\perp.$$

THÉORÈME 12. *Soient Ω un ouvert de R^N, $X \subset L^2(\Omega)$ un espace de Hilbert et $L : X \to X$ un opérateur linéaire auto-adjoint tel qu'il existe un réel positif ρ pour lequel $L[u] + \rho u$ est elliptique. Supposons que $\mu_n \in \mathbb{R}$ est une valeur propre de L tel que l'ensemble X_n des fonctions propres associées vérifient la condition* (38)*. Si g une fonction continue, périodique de moyenne nulle et $h \in X_n^\perp$, alors la fonctionnelle I_0 définie par* (37) *possède une valeur critique c. De plus,*
(i) s'il existe $w_0 \in X_n$ tel que

$$\max_{e \in X_-} \left\{ \int_\Omega G(\bar{u}(x) + w_0(x) + e(x))\, dx - \frac{(\mu_n - \mu_{n-1})}{2} \|e\|_2^2 \right\} < 0, \quad (39)$$

alors $c < \bar{I}(\bar{u})$ et il existe $\zeta > 0$ (qui dépend de h) tel que la fonctionnelle I_w possède un point critique pour tout $w \in X_n$ satisfaisant $\|w\| < \zeta$;
(ii) s'il existe $w_0 \in X_n$ tel que

$$\max_{e \in X_-} \left\{ \int_\Omega G(\bar{u}(x) + w_0(x) + e(x))\, dx - \frac{(\mu_n - \mu_{n-1})}{2} \|e\|_2^2 \right\} = 0, \quad (40)$$

alors, l'une des deux situations suivantes a lieu :
(a) *$c = \bar{I}(\bar{u})$ et I_0 a un point critique de la forme $\bar{u} + w_0 + e$, où $e \in X_-$ maximise* (40) *;*
(b) *$c < \bar{I}(\bar{u})$ et il existe $\zeta > 0$ (dépendant de h) tel que la fonctionnelle I_w possède un point critique pour tout $w \in X_n$ tel que $\|w\| < \zeta$.*

La preuve de ce théorème s'effectue en deux étapes. Dans un premier temps, nous montrons que la fonctionnelle I_0 a une géométrie de type point de selle à un niveau $c \le \bar{I}(\bar{u})$. La fonctionnelle I_w est ensuite considérée comme une petite perturbation de I_0 et nous observons que la géométrie de point de selle se maintient tant que w n'est pas trop grand. Pour décrire la géométrie de I_0, notons $D \subset \bar{u} + w_0 + X_-$ la boule fermée de centre $\bar{u} + w_0$ et de rayon $R > 0$. Lorsque R est suffisamment grand, l'inégalité

$$I_0(u) < \inf_{\bar{u}+X_-^\perp} I_0$$

est satisfaite sur le bord ∂D de D. Dès lors, le niveau de minimax

$$c := \inf_{\gamma \in \Gamma} \max_{u \in D} I_0(\gamma(u)),$$

où $\Gamma := \{\gamma \in C(D, X) \mid \gamma_{|\partial D} = id_{|\partial D}\}$, est un niveau critique de I_0 si une suite de Palais-Smale au niveau c possède une sous-suite convergente. La condition (39) suffit pour montrer que $c < \bar{I}(\bar{u})$ ce qui permet de conclure. Sous l'hypothèse (40), il se peut que $c = \bar{I}(\bar{u})$. Dans ce cas l'infimum c est en réalité un minimum atteint par le choix particulier de $\gamma = i_D : D \to X$.

Pour définir un niveau de minimax associé à I_w, l'argument invoqué pour obtenir un point critique de I_0 n'est plus valable. En effet, dès que $w \neq 0$, nous avons $\inf_{\bar{u}+X_-^\perp} I_w = -\infty$. Cependant, si C est une boule dans $\bar{u} + X_-^\perp$ centrée en $\bar{u}$, alors $\inf_C I_w > c$ lorsque w est suffisamment petit tandis que $\sup_{\partial D} I_w < c$ si R est assez grand. Nous avons donc mis en évidence une géométrie de minimax à condition que la classe des déformations choisies pour calculer la valeur de minimax satisfasse la condition d'intersection avec la boule C. Pour cela, nous choisissons une classe de déformations plus petite que précédemment, suivant un argument utilisé par N. Ghoussoub [G].

Sous des hypothèses supplémentaires, la fonctionnelle I_w possède de nombreux points critiques. Commençons par énoncer un résultat de multiplicité concernant I_0.

Théorème 13. *Soient Ω un ouvert de R^N, $X \subset L^2(\Omega)$ un espace de Hilbert et $L : X \to X$ un opérateur linéaire auto-adjoint tel qu'il existe un réel positif ρ pour lequel $L[u]+\rho u$ est elliptique. Supposons que $\mu_n \in \mathbb{R}$ est une valeur propre de L tel que l'ensemble X_n des fonctions propres associées vérifient la condition* (38). *Soient g une fonction continue, périodique de moyenne nulle et $h \in X_n^\perp$. Supposons qu'il existe $\xi > 0$ et une suite d'ensembles compacts $\mathcal{C}_n \subset X_n$, $n \in \mathbb{N}$ tels que :*
(a) pour tout $w \in \mathcal{C}_n$ et tout $v \in X_+$ satisfaisant $\|v\| \leq \xi$,

$$\int_\Omega G(\bar{u}(x) + w(x) + v(x))\, dx \geq 0\,;$$

(b) la composante connexe non bornée $\mathcal{F}_n$ de $X_n \setminus \mathcal{C}_n$ est telle que

$$\min_{w \in \mathcal{F}_n} \|w\| \to +\infty \text{ pour } n \to +\infty.$$

Si de plus, il existe une suite $w_n \in X_n$, $n \in \mathbb{N}$ telle que $\|w_n\| \to +\infty$ et la condition (39) *est satisfaite, avec w_0 remplacé par w_n, alors la fonctionnelle I_0 possède une suite de points critiques $(u_i)_i \subset X$ telle* *que* $\|u_i\| \to +\infty$ *et la suite de niveaux critiques* $c_i = I_0(u_i)$ *converge de manière strictement croissante vers* $\bar{I}(\bar{u})$*.*

L'idée principale de la démonstration de ce résultat est de construire une infinité de niveaux de minimax en choisissant une suite de boules D_n contenues dans $\bar{u}+w_n+X_-$ et centrées en $\bar{u}+w_n$. En appliquant le même argument que précédemment, nous obtenons une suite de points critiques. Les ensembles compacts $\mathcal{C}_n$ constituent alors des barrières qui "emprisonnent" les déformations le long du flot gradient, ce qui permet d'assurer l'existence d'une sous-suite de points critiques distincts.

La multiplicité des points critiques se maintient pour I_w lorsque la norme de w est suffisamment petite.

THÉORÈME 14. *Sous les hypothèses du théorème précédent, pour tout* $n \in \mathbb{N}$*, il existe* $\varepsilon_n > 0$ *tel que* I_w *possède* n *points critiques si* $w \in X_n$ *satisfait* $\|w\| < \varepsilon_n$*.*

Applications. Venons-en aux applications à quelques problèmes concrets. Le premier problème sur lequel nous avons testé ces résultats abstraits est le problème 2π-périodique associé à l'équation (35). Pour pouvoir appliquer le Théorème 12, nous devons trouver $w_0 = A\cos(t-\delta)$ dans $X_2 := \langle \sin(\cdot), \cos(\cdot)\rangle$ tel que

$$\max_{e\in\mathbb{R}} \left\{ \int_0^{2\pi} \cos(\bar{u}(t) + A\cos(t-\delta) + e)\,dt - 2\pi^2 e^2 \right\} \le 0, \tag{41}$$

où $\bar{u}$ est la solution 2π-périodique de l'équation

$$u''(t) + u(t) = h(t)$$

qui se trouve dans $X_2^{\perp}$. L'intégrale figurant dans la condition (41) peut être estimée grâce à *la méthode de la phase stationnaire* [E] lorsque $A \to +\infty$. Cette estimation permet d'obtenir l'inégalité stricte dans (41) pour une valeur A suffisamment grande lorsque $\bar{u}$ satisfait la condition

$$\bar{u}(\delta) + \bar{u}(\delta+\pi) \ne (2n+1)\pi,\ n \in N, \tag{42}$$

pour un certain $\delta \in [0, 2\pi]$. Dans le cas contraire, l'estimation ne permet que d'obtenir l'égalité dans (41) pour une certaine valeur de A. Un argument supplémentaire nous assure alors que c'est l'affirmation (ii)-(b) du Théorème 12 qui est vraie.

Pour appliquer le résultat de multiplicité, nous supposons que l'hypothèse (42) est satisfaite pour tout $\delta \in [0, 2\pi[$. Dans ce cas, l'estimation obtenue grâce à la méthode de la phase stationnaire permet d'obtenir une suite non bornée de courbes $\mathcal{C}_n = \{A_n(\delta),\ \delta \in [0, 2\pi[\}$ satisfaisant les hypothèses du Théorème 13.

Mentionnons encore quelques applications traitées dans [BFR]. Le résultat d'existence s'étend au cas du problème 2π-périodique associé à l'équation (35), où la fonction $\sin u$ est remplacée par une fonction g de classe C^1 et de valeur moyenne nulle.

Nous traitons également le problème périodique à la troisième valeur propre

$$u'' + 4u + \sin u = h(t) + a\sin(2(t-\phi))$$

et le problème de Dirichlet sur $[0,\pi]$ associé à l'équation de Sturm-Liouville

$$(p(t)u')' + q(t)u + \alpha\sin(u(t)) = h(t) + a\phi(t).$$

Notre approche nécessite dans ces deux derniers cas des hypothèses supplémentaires.

Enfin, nous testons le Théorème 12 sur une équation aux dérivées partielles. Nous considérons le problème de Dirichlet

$$\begin{aligned} \Delta u + 5u - \sin u &= h(x,y) + a_1\sin 2x\sin y + a_2\sin x\sin 2y && \text{dans } \Omega,\\ u &= 0 && \text{sur } \partial\Omega, \end{aligned} \tag{43}$$

où $\Omega = [0,\pi]\times[0,\pi]$, $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$ et la fonction $h \in L^2(\Omega)$ est orthogonale aux fonctions $\sin 2x\sin y$ et $\sin x\sin 2y$. Nous obtenons au moins une solution lorsque a_1 et a_2 sont suffisamment petits. La preuve requiert une adaptation de la méthode de la phase stationnaire en dimension 2 pour des fonctions prenant des valeurs critiques sur le bord du domaine.

Solutions radiales de problèmes elliptiques et EDOs singulières

Il est bien connu que les solutions radiales $v(x) := u(|x|)$ de l'équation de Laplace

$$\Delta v = g(v) \tag{44}$$

dans une boule (ou tout $\mathbb{R}^N$) satisfont, le long d'un rayon, l'équation différentielle singulière

$$u'' + (N-1)\frac{u'}{t} = g(u). \tag{45}$$

Inversement, les solutions de cette dernière équation qui vérifient la condition $u'(0) = 0$ donnent lieu à des solutions radiales de l'équation (44). L'équation de Laplace occupe une place de choix dans de nombreux problèmes de physique mathématique, et l'existence de solutions radiales positives pour cette équation a suscité bon nombre de travaux. Parmi les plus classiques, citons les contributions de S. I. Pohozaev [P], Z. Nehari [N], H. Berestycki, P. L. Lions et L. A. Peletier [BLP] et H. Berestycki et P. L. Lions [BL]. L'article qui a retenu notre attention est celui de H. Berestycki, P. L. Lions et L. A. Peletier. Comme nous venons de le mentionner, l'existence de solutions radiales de l'équation (44) est un problème essentiellement unidimensionnel. L'approche de

H. Berestycki et al. se base effectivement sur l'existence de solutions de l'équation singulière (45) mais curieusement, un de leurs arguments nécessite un détour par la théorie des équations aux dérivées partielles et la résolution d'un problème elliptique dont on sait que la solution est radiale grâce à un résultat classique de symétrie de B. Gidas, W. M. Ni et L. Nirenberg [GNN]. Ceci nous a motivé à trouver un argument alternatif de nature "EDO". En particulier, nous cherchons un argument valable même lorsque le paramètre N dans (45) n'est pas entier. Dans un travail en collaboration avec J. M. Gomes et L. Sanchez [BGS], nous étudions les solutions positives du problème aux limites

$$\begin{gathered} u'' + k\frac{u'}{t} = c(t)g(u), \\ u'(0) = 0, \ u(M) = 0, \end{gathered} \tag{46}$$

où $0 < M \le \infty$ et $k > 1$. Lorsque k est un entier, les solutions de (46) étendues par symétrie radiale à la boule de rayon M, ou à $\mathbb{R}^N$ si $M = +\infty$, sont solutions du problème de Dirichlet pour l'équation de Laplace correspondante

$$\Delta u = c(|x|)g(u).$$

Nous supposons que les fonctions c et g satisfont les hypothèses suivantes. La fonction c est positive et continue dans $[0,\infty[$ et g est une fonction localement lipschitzienne dans $[0,\infty[$ telle que $g(0) = 0$. En outre, nous supposons que g est positive sur un certain intervalle $]0,a[$, négative sur $]a,+\infty[$ et que sa primitive $G(u) = \int_0^u g(s)\,ds$ prend des valeurs négatives.

Ces hypothèses sont adéquates pour traiter le problème de Dirichlet (46) lorsque g est une fonction bornée. Lorsque ce n'est pas le cas, notre approche nécessite une restriction de croissance à l'infini

$$\limsup_{u\to+\infty} \frac{|g(u)|}{u^r} < +\infty$$

avec $r \in]0, \frac{k+3}{k-1}[$. Un bon modèle est donné par $g(u) = \alpha u - \beta u^p$ avec α, $\beta > 0$ et $1 < p < \frac{k+3}{k-1}$. La valeur $\frac{k+3}{k-1}$ joue en fait le rôle d'exposant critique de Sobolev. Enfin, il est pratique de supposer que $g(u) = 0$ pour $u \le 0$. En effet, cela implique que les solutions non triviales de (46) sont strictement positives.

Nonlinéarités bornées. Le problème de Dirichlet (46) possède une structure variationnelle dans un espace de Sobolev à poids. Nous commençons par supposer $M < +\infty$. Les solutions de (46) correspondent aux points critiques de la fonctionnelle

$$J_M(u) := \int_0^M t^k \left(\frac{u'(t)^2}{2} + c(t)G(u(t)) \right) dt \tag{47}$$

dans l'espace de Hilbert

$$H_k(0,M) := \left\{ u \in W^{1,1}(0,M) \mid u(M) = 0 \text{ et } \int_0^M t^k u'(t)^2\,dt < +\infty \right\}$$

muni de la norme

$$\|u\|_M := \sqrt{\int_0^M t^k u'(t)^2\,dt}.$$

Pour rappel, l'espace $W^{1,1}(0,M)$ est l'ensemble des fonctions continues sur $[0,M]$ dont la dérivée faible est intégrable.

Proposition 15. *Soient* $0 < M < +\infty$ *et* $k > 1$. *Supposons que*

(a) *c est une fonction continue dans $[0,\infty[$ et il existe deux nombres réels c_1 et c_2 tels que $0 < c_1 \le c(t) \le c_2$ pour tout $t \in [0,\infty[$;*

(b) *g est localement lipschitzienne dans $[0,\infty[$ et $g(0)=0$;*

(c) *il existe $a > 0$ tel que*

$$(a-u)g(u) > 0 \quad si \quad u \in \mathbb{R}^+ \setminus \{a\}$$

et $\xi > a$ tel que

$$G(\xi) = \int_0^\xi g(s)\,ds < 0;$$

(d) *il existe $r \in \left]0, \frac{k+3}{k-1}\right[$ tel que*

$$\limsup_{u\to+\infty} \frac{|g(u)|}{u^r} < +\infty.$$

Alors, nous avons les résultats suivants :

(i) La fonctionnelle J_M définie par (47) *est de classe C^1 et son gradient peut se décomposer en $J'_M(u) = u + N(u)$, où $N : H_k(0,M) \to H_k(0,M)$ est un opérateur compact.*

(ii) La fonctionnelle J_M est faiblement semi-continue inférieurement dans l'espace fonctionnel $H_k(0,M)$. Si de plus g est bornée, alors J_M est coercitive.

(iii) Les points critiques de J_M dans $H_k(0,M)$ sont des solutions de (46) *de classe C^1 dans $[0,M]$.*

La preuve de cette proposition est standard sauf en ce qui concerne le point (iii). La difficulté consiste à montrer que les points critiques ont une dérivée nulle en 0. Cette proposition repose en fait essentiellement sur les injections

$$H_k(0,M) \hookrightarrow L^2_s(0,M) \quad \text{si} \quad s > k-2,$$

$$H_k(0,M) \hookrightarrow L^q_k(0,M) \quad \text{si} \quad 2 \le q < \frac{2k+2}{k-1},$$

où $L^q_k(0,M)$ désigne l'espace de Lebesgue à poids

$$L^q_k(0,M) := \{u : (0,M) \to \mathbb{R} \text{ mesurable } \mid \int_0^M t^k u(t)^q \, dt < +\infty\}.$$

Ces injections, qui sont bien connues lorsque k est entier, sont en outre compactes. Une première conséquence évidente se déduit du point (ii) de la Proposition 15. Si g est une fonction bornée, la fonctionnelle J_M atteint un minimum dans $H_k(0,M)$. Pour M petit, l'unique minimant est la solution triviale tandis que pour M assez grand il est aisé de vérifier que J_M prend des valeurs négatives assurant donc que les minimants sont des fonctions positives. D'autre part, l'injection de $H_k(0,M)$ dans $L^q_k(0,M)$ pour $2 \le p \le \frac{2k+2}{k-1}$ implique que 0 est un minimum local strict de J_M dans $H_k(0,M)$. En effet, puisque g est bornée, il existe $p \in]2, \frac{2k+2}{k-1}[$ et $L > 0$ tel que

$$G(u) \ge -L|u|^p$$

pour tout $u \in \mathbb{R}$. On en déduit directement que

$$J_M(u) \ge \frac{1}{2}\|u\|^2_M - C\|u\|^p_M.$$

Pour simplifier la présentation de notre résultat principal, nous supposons dès à présent que $c(t) \equiv 1$. Un résultat légèrement moins précis est démontré dans [BGS] pour une fonction c non constante.

Théorème 16. *Soient $k > 1$ et g une fonction bornée vérifiant les hypothèses* (b) *et* (c) *de la Proposition* 15. *Si $c(t) \equiv 1$, il existe $M_0 > 0$ tel que le problème* (46) *possède au moins deux solutions positives si $M > M_0$, au moins une solution positive si $M = M_0$ et aucune solution positive si $M < M_0$.*

L'idée principale est de montrer que $J_{M'}$ compte au moins deux points critiques non triviaux pour $M' > M$ dès que J_M en possède un différent de 0. Il suffit ensuite de définir

$$M_0 := \inf\{M \mid J_M \text{ possède un point critique non trivial}\}.$$

Il est clair que $M_0 < +\infty$ puisque nous avons déjà observé que le minimant de J_M est une fonction positive si M est suffisamment grand. D'autre part, les hypothèses faites sur g et G impliquent l'existence de $\bar{a} > a$ tel que $G(\bar{a}) = 0$. Si u est une solution de (46) avec $u(0) \in]0, \bar{a}[$, alors $u(t) \neq 0$ pour tout $t > 0$. En effet, en multipliant l'équation par u' et en intégrant, on s'aperçoit que

$$G(u(t)) \ge G(u(0)).$$

On en déduit que si u est solution de (46) pour $M > 0$, $u(0) \ge \bar{a}$. Par ailleurs, il est facile de vérifier que $u(t) \ge u(0) - Ct^2$ pour un certain C (dépendant de g mais pas de u) si bien que u ne peut pas être solution de (46) si M est trop petit. On en conclut que $M_0 > 0$. Pour $M = M_0$, un argument standard permet d'obtenir une solution par approximation pour $M \searrow M_0$.

Revenons à l'argument principal qui mène à l'existence de deux solutions pour $M > M_0$. Si v désigne un point critique de J_{M_0}, notons w son extension continue par 0 sur $[0, M]$ pour $M > M_0$ et

$$\mathcal{C} := \{u \in H_k(0, M) \mid u(t) \geq w(t) \text{ pour tout } t \in [0, M]\}\,.$$

Il est clair que $\mathcal{C}$ est un sous-ensemble convexe fermé de $H_k(0, M)$ et dès lors J_M atteint un minimum dans $\mathcal{C}$. Cela n'implique pas pour autant que les minimants dans $\mathcal{C}$ soient des points critiques. Pour pouvoir faire cette dernière affirmation, il faut s'assurer que les minimants appartiennent à $\operatorname{int}\mathcal{C} = \{u \in \mathcal{C} \mid u(t) > w(t) \text{ pour tout } t \in [0, M]\}$. Ceci se prouve par contradiction. Si u est un minimant de J_M dans $\mathcal{C}$ tel que $u(t) = w(t)$ pour un certain $t \in]0, M[$ il existe une fonction test h (dont le support est un petit intervalle centré en t) telle que $u + h \in \mathcal{C}$ et $J'_M(u)(h) < 0$. Le long du chemin $s : [0, 1] \to \mathcal{C}, s \mapsto u + sh$, J décroît donc dans un voisinage de $s = 0$ contredisant la définition de u. Nous pouvons maintenant conclure que les minimants dans $\mathcal{C}$ sont des points critiques de J_M et donc des solutions de (46).

La deuxième solution s'obtient grâce au *Mountain-Pass theorem* de A. Ambrosetti et P. H. Rabinowitz [AR], voir aussi [R]. En effet, il s'avère que le minimum de J_M dans $\mathcal{C}$ est en plus un minimum local dans $H_k(0, \infty)$. Comme la fonctionnelle J_M possède également un minimum local en 0, elle présente une géométrie du type Mountain-Pass. Par ailleurs, comme g est bornée, il est aisé de vérifier que les suites de Palais-Smale sont bornées dans $H_k(0, M)$. Le point (i) de la Proposition 15 permet alors d'en extraire une sous-suite convergente. Nous avons donc tous les ingrédients pour appliquer le Théorème du Mountain-Pass de A. Ambrosetti et P. H. Rabinowitz. Le point critique obtenu est une seconde solution positive de (46).

Nous n'avons jusqu'à présent considéré que le cas $M < +\infty$. Le problème sur la demi-droite peut en fait se résoudre facilement grâce à un argument topologique astucieux de H. Berestycki, P. L. Lions et L. A. Peletier [BLP]. L'idée est de considérer le problème de Cauchy

$$u'' + k\frac{u'}{t} = c(t)g(u), \quad u(0) = \eta,\ u'(0) = 0,$$

où $\eta > 0$. Des résultats classiques de la théorie des EDOs permettent d'affirmer que ce problème possède une solution unique

$$u(\cdot, \eta) \in C^1([0, \infty[) \cap C^2(]0, \infty[)$$

continue en la donnée initiale η. Considérons l'intervalle $I =]a, \infty[$ et les deux sous-ensembles : I_+, contenant les valeurs $\eta \in I$ telles que la solution $u(t, \eta)$ possède un minimum local positif en $t_0 > 0$ et $u'(t, \eta) < 0$

pour $0 < t < t_0$ et I_-, contenant les valeurs $\eta \in I$ telles que la solution $u(t,\eta)$ s'annule en un certain $M > 0$ et $u'(t,\eta) < 0$ pour $0 < t \leq M$. Il est assez simple de prouver que I_+ et I_- sont deux ensembles ouverts disjoints [BLP]. Lorsque $g'(a) < 0$, nous prouvons dans [BGS] qu'ils sont non vides ; I_+ contient des valeurs $\eta > a$ proches de a, tandis que les minimants de J_M ont leurs valeurs initiales dans I_- lorsque M est assez grand. La connexité de I implique l'existence d'une valeur initiale $\eta \in I \setminus (I_+ \cup I_-)$. La solution correspondante est strictement décroissante et tend vers 0 à l'infini. Nous pouvons donc énoncer le résultat suivant.

Théorème 17. *Soit $k > 1$. Supposons que c est une fonction continue satisfaisant l'hypothèse* (a) *de la Proposition* 15 *et que g est une fonction bornée vérifiant les conditions* (b) *et* (c) *de la Proposition* 15. *Si de plus, g est dérivable au point a et $g'(a) < 0$, alors le problème* (46) *avec $M = +\infty$ possède une solution u telle que $u'(t) < 0$ pour tout $t > 0$.*

Du point de vue variationnel, cette solution peut s'obtenir comme un point critique de la fonctionnelle

$$J_\infty(u) := \int_0^\infty t^k \left(\frac{u'(t)^2}{2} + c(t)G(u(t)) \right) dt$$

dans l'espace

$$H_k(0,\infty) := \left\{ u \in H^1_{\mathrm{loc}}(0,\infty) \middle| \int_0^\infty t^k \left(u(t)^2 + u'(t)^2 \right) dt < +\infty \right\}$$

muni de la norme

$$\|u\| := \sqrt{\int_0^\infty t^k \left(u(t)^2 + u'(t)^2 \right) dt}.$$

Le niveau critique de J_M qui survit lorsque $M \to +\infty$ est le niveau de Mountain-Pass. Le minimum dans $\mathcal{C}$, lui, disparaît dans le passage à la limite.

Nonlinéarités non bornées. Lorsque g est non bornée, notre approche nécessite des hypothèses renforcées et nos résultats sont moins précis. En plus de la condition de croissance à l'infini que nous avons énoncée précédemment, nous supposons qu'il existe $\theta \in]0, \frac{1}{2}[$ et $\rho \in \mathbb{R}$ tels que

$$G(u) - \theta u g(u) \geq \rho \tag{48}$$

pour tout $u \geq 0$. Cette condition dite de “superlinéarité” a été introduite par A. Ambrosetti et P. H. Rabinowitz [AR,R]. Elle permet de démontrer que les suites de Palais-Smale sont bornées dans $H_k(0,M)$. Concernant le problème sur l'intervalle borné $[0,M]$, nous démontrons le théorème suivant.

THÉORÈME 18. *Soient $k > 1$ et $M < +\infty$. Supposons que c et g vérifient les hypothèses* (a), (b), (c) *et* (d) *de la Proposition* 15. *Si de plus g satisfait la condition de superlinéarité* (48) *avec $\theta \in]0, \frac{1}{2}[$ et $\rho \in \mathbb{R}$, alors le problème* (46) *possède au moins une solution positive lorsque M est suffisamment grand.*

La preuve se base également sur une géométrie de Mountain-Pass. L'origine est un minimum local et pour M assez grand, il existe des fonctions de $H_k(0, M)$ qui donnent une valeur négative de J_M.

Lorsque $M = +\infty$, nous abordons le problème par approximation sur des intervalles bornés.

THÉORÈME 19. *Soient $k > 1$, $\alpha > 0$ et h une fonction continue. Supposons que c est une fonction vérifiant l'hypothèse* (a) *de la Proposition* 15*, $g(u) = \alpha u + h(u)$ satisfait les conditions* (b), (c) *de la Proposition* 15 *et l'hypothèse de superlinéarité* (48) *pour $\rho = 0$. Supposons de plus que h est tel que pour tout $u \geq 0$,*

$$|h(u)| \leq Lu^q,$$

pour certaines constantes $L > 0$ et $q \in]1, \frac{k+3}{k-1}[$. Alors le problème (46) *avec $M = +\infty$ possède au moins une solution positive.*

La solution est obtenue comme limite faible dans $H_k(0, \infty)$ d'une suite $(u_m)_m$ de solutions de (46) pour $M = m \to +\infty$ étendues par 0 en dehors de $[0, m]$. La restriction de croissance sur h permet d'obtenir une injection de $H_k(0, \infty)$ dans un "bon" espace tandis que la condition de superlinéarité implique une borne uniforme sur les niveaux de Mountain-Pass des solutions approximantes. Cette borne à son tour livre de précieuses estimations sur la suite u_m permettant d'en extraire une sous-suite faiblement convergente.

Solutions positives d'EDPs elliptiques superlinéaires indéfinies

Le dernier problème que j'ai étudié concerne une équation aux dérivées partielles. Nous avons considéré, avec J. M. Gomes et P. Habets, le problème superlinéaire indéfini

$$\begin{aligned} \Delta u + (a_+(x) - \mu a_-(x))|u|^\gamma u &= 0 \ \text{ dans } \Omega, \\ u(x) &= 0 \ \text{ sur } \partial\Omega, \end{aligned} \tag{49}$$

où $\Omega \subset \mathbb{R}^N$ est un domaine borné de classe $\mathcal{C}^1$, a_+ et a_- sont des fonctions continues, γ est un exposant strictement positif et μ est un paramètre réel positif. Nous supposons que l'exposant $\gamma + 2$ est sous-critique c.-à-d. tel que $\gamma + 2 < 2^* = \frac{2N}{N-2}$ si $N \geq 3$. Cette hypothèse classique assure l'injection compacte de $H_0^1(\Omega)$ dans $L^{\gamma+2}(\Omega)$.

La fonction $a_\mu := a_+ - \mu a_-$ est appelée "poids". Nous supposons que ce poids change de signe dans Ω. C'est pour cela que le problème est qualifié d'indéfini. Plus précisément, nous faisons l'hypothèse que le domaine Ω est partitionné en deux sous-domaines $\Omega_+ := \bigcup_{i=1}^n \omega_i$, $\Omega_- := \Omega \backslash \overline{\Omega}_+$ de classe C^1 tels que

(a) pour tout $x \in \Omega_+$, $a_-(x) = 0$, $a_+(x) > 0$,

(b) pour tout $x \in \Omega_-$, $a_-(x) > 0$, $a_+(x) = 0$ et

(c) pour tout $i \neq j$, $\omega_i \cap \omega_j = \emptyset$.

La question d'existence de solutions positives pour des problèmes indéfinis tels que (49) a préoccupé bon nombre de spécialistes des EDPs ces dernières années. Diverses approches peuvent être utilisées pour aborder cette question, les principales étant les méthodes topologiques et variationnelles [AT], [AL], [BCN], [Lo].

L'existence d'une solution positive du problème de Dirichlet (49) n'est pas une question difficile. En effet, les solutions positives de (49) sont des points critiques de la fonctionnelle $I : H_0^1(\Omega) \to \mathbb{R}$ définie par

$$I(u) := \int_\Omega \left(\frac{1}{2} |\nabla u(x)|^2 - (a_+(x) - \mu a_-(x)) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \right) dx.$$

Étant donné que $\gamma > 0$, il est bien connu que l'inégalité de Poincaré implique que l'origine est un minimum local de I. D'autre part, il est très simple de vérifier que la fonctionnelle I n'est pas minorée. Cette dernière présente donc une géométrie de Mountain-Pass et la condition de sous-criticité de l'exposant $\gamma + 2$ donne de la compacité suffisante aux suites de Palais-Smale. Nous déduisons donc immédiatement l'existence d'un point critique non trivial du Théorème du Mountain-Pass. La question qui nous intéresse est en fait celle de l'existence de plusieurs solutions lorsque le paramètre μ est grand.

L'homogénéité du terme non linéaire permet d'obtenir les solutions de (49) d'une façon alternative en considérant la fonctionnelle d'énergie $E : H_0^1(\Omega) \to \mathbb{R}$ définie par

$$E(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx \tag{50}$$

restreinte à la variété

$$\mathcal{V}_\mu := \left\{ u \in H_0^1(\Omega) \mid \int_\Omega (a_+(x) - \mu a_-(x)) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx = 1 \right\}. \tag{51}$$

La condition $V_\mu(u) = 1$, où $V_\mu(u) : H_0^1(\Omega) \to \mathbb{R}$ est la fonctionnelle définie par

$$V_\mu(u) := \int_\Omega (a_+(x) - \mu a_-(x)) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx,$$

est généralement appelée *la contrainte.*

En utilisant l'injection de $H^1_0(\Omega)$ dans $L^{\gamma+2}(\Omega)$, il est assez direct de vérifier que la variété $\mathcal{V}_\mu$ est faiblement fermée et que la fonctionnelle $V_\mu : H^1_0(\Omega) \to \mathbb{R}$ est de classe $C^{1,1}$. Dès lors, les points critiques de E sous contrainte vérifient l'équation d'Euler-Lagrange

$$\nabla E(u) = \lambda \nabla V_\mu(u)$$

pour un certain *multiplicateur de Lagrange* $\lambda \in \mathbb{R}$, c.-à-d.

$$\int_\Omega \nabla u(x) \nabla w(x)\, dx = \lambda \int_\Omega (a_+(x) - \mu a_-(x)) u_+^{\gamma+1}(x) w(x)\, dx \qquad (52)$$

pour tout $w \in H^1_0(\Omega)$. Il s'ensuit que les point critiques satisfont l'EDP

$$\Delta u + \lambda(a_+(x) - \mu a_-(x)) u_+^{\gamma+1} = 0.$$

Par ailleurs, en choisissant les points critiques comme fonctions tests dans (52), nous obtenons

$$\lambda = \frac{1}{\gamma+2} \int_\Omega |\nabla u(x)|^2\, dx > 0$$

puisque $0 \notin \mathcal{V}_\mu$. La fonction renormalisée $v = \lambda^{1/\gamma} u$ est donc une solution de (49).

Définissons également, pour $i = 1, ..., n$, les variétés

$$\hat{\mathcal{V}}_i := \{u \in \mathcal{V}_\mu \mid \operatorname{supp} u \subset \overline{\omega}_i\}. \qquad (53)$$

Il est clair que E est coercitive sur $\mathcal{V}_\mu$ et faiblement semi-continue inférieurement si bien que E atteint un minimum dans chaque variété $\hat{\mathcal{V}}_i$. Notons $\hat{u}_i$ les minimants respectifs. Chaque minimant renormalisé est une solution positive de l'équation

$$\Delta u + (a_+(x) - \mu a_-(x))|u|^\gamma u = 0$$

dans ω_i s'annulant sur son bord.

Lorsque le paramètre μ est grand, les fonctions de $\mathcal{V}_\mu$ d'énergie finie doivent être petites dans Ω_- étant donnée la contrainte

$$\int_{\Omega_-} a_-(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2}\, dx = \frac{1}{\mu} \left(\int_{\Omega_+} a_+(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2}\, dx - 1 \right).$$

Si ce sont de plus des points critiques, comme ils sont proches de 0 sur les bords des sous-domaines ω_i, on s'attend à ce qu'ils soient proches sur chaque ω_i d'une solution du problème de Dirichlet dans ω_i (par exemple proches des minimants dans $\hat{\mathcal{V}}_i$). Pour l'EDO équivalente à (49), M. Gaudenzi, P. Habets et F. Zanolin [GHZ1,GHZ2] ont obtenu ce genre de solutions lorsque le poids est positif sur deux ou trois intervalles disjoints. Ils démontrent respectivement l'existence, pour μ suffisamment grand, de trois et sept solutions à l'aide d'une méthode de tir. Sur un intervalle où le poids est positif, les solutions sont soit proches d'une solution positive du problème de Dirichlet sur cet intervalle soit proche de 0.

Étant données ces observations, nous conjecturons qu'il existe, pour μ assez grand, $2^n - 1$ solutions positives de (49) qui se concentrent sur un certain nombre de sous-domaines ω_i et qui sont proches de 0 ailleurs. Bien que certaines solutions soient faciles à mettre en évidence, une des difficultés majeures consiste à faire la distinction entre toutes ces solutions. Pour cela, nous introduisons la définition de *solution p-bosses*. Grossièrement, il s'agit d'une solution qui n'a de contributions significatives à la contrainte que dans p sous-domaines ω_i. Cette description étant assez vague, il est préférable de définir la notion de famille de solutions p-bosses à travers les propriétés des fonctions limites obtenues lorsque $\mu \to +\infty$.

Définitions 4. Soit $1 \leq p \leq n$. Une fonction $w \in H_0^1(\Omega)$ est une *p-bosses* si $w = e_1 + \ldots + e_p$, où les p fonctions non nulles $e_j \in H_0^1(\Omega)$ sont telles que $\operatorname{supp} e_j \subset \overline{\omega}_{i_j}$, $i_j \in \{1, \ldots, n\}$ et $i_j \neq i_k$ pour $j, k \in \{1, \ldots, p\}$.

Si $\omega = \omega_{i_1} \cup \ldots \cup \omega_{i_p}$, l'ensemble de fonctions $\{u_\mu | \mu \geq \mu_0\} \subset H_0^1(\Omega)$ est appelé *famille de solutions p-bosses de* (49) *à support dans $\overline{\omega}$* si pour tout $\mu \geq \mu_0$, u_μ est une solution de (49), l'ensemble possède une valeur d'adhérence pour la topologie faible dans $H_0^1(\Omega)$ pour $\mu \to +\infty$, et toute valeur d'adhérence est une p-bosses $w \in H_0^1(\Omega)$ dont le support est inclus dans $\overline{\omega}$.

Nous démontrons dans notre contribution [BGH] le résultat de multiplicité suivant.

Théorème 20. *Soit $\Omega \subset \mathbb{R}^N$ un domaine borné de classe $\mathcal{C}^1$, partitionné en deux sous-domaines $\Omega_+ = \cup_{i=1}^n \omega_i$, $\Omega_- = \Omega \backslash \overline{\Omega}_+$ de classe C^1 tels que pour tout $i \neq j$, $\omega_i \cap \omega_j = \emptyset$. Soient a_+ et a_- deux fonctions continues telles que pour tout $x \in \Omega_+$, $a_-(x) = 0$, $a_+(x) > 0$ et pour tout $x \in \Omega_-$, $a_-(x) > 0$, $a_+(x) = 0$. Soit $\gamma > 0$ tel que $\gamma + 2 < \frac{2N}{N-2}$ si $N \geq 3$. Pour tout ensemble $\omega = \omega_{i_1} \cup \ldots \cup \omega_{i_p}$, il existe $\mu_\omega > 0$ et une famille $\{u_\mu \mid \mu \geq \mu_\omega\}$ de solutions p-bosses positives de* (49) *à support dans $\overline{\omega}$. En particulier, le problème de Dirichlet* (49) *possède au moins $2^n - 1$ solutions positives pour μ suffisamment grand.*

L'approche que nous avons choisie pour démontrer l'existence de ces solutions requiert un certain nombre de résultats "techniques" que nous passons sous silence pour ne garder que les idées principales. Nous obtenons les solutions du Théorème 20 comme points critiques de E dans $\mathcal{V}_\mu$. Puisque nous prétendons prouver l'existence de plusieurs solutions, il faut tout d'abord s'assurer que deux points critiques différents de E sous contrainte mènent à des solutions différentes. Ceci se déduit facilement de la façon dont se fait la renormalisation. Comme nous l'avons déjà mentionné, la sous-criticité de l'exposant $\gamma + 2$ donne suffisamment de compacité aux suites de Palais-Smale, pour la différentielle de I, en vue d'appliquer les théorèmes classiques de la théorie des points critiques. Il en est de même pour le gradient de E projeté dans les espaces tangents à $\mathcal{V}_\mu$.

Les solutions que nous cherchons peuvent s'écrire comme la somme de fonctions multi-bosses et d'une petite perturbation. Ceci suggère une décomposition adéquate de $H_0^1(\Omega)$ en une somme orthogonale $\bar{H} \oplus \tilde{H}$, où

$$\bar{H} := \{u \in H_0^1(\Omega) \mid \operatorname{supp} u \subset \bar{\Omega}_+\}$$

et

$$\tilde{H} := (\bar{H})^\perp = \{u \in H_0^1(\Omega) \mid \int_\Omega u\, v\, dx = 0 \text{ pour tout } v \in \bar{H}\}.$$

L'ensemble $\bar{H}$ est l'espace des fonctions multi-bosses. Cette décomposition possède de bonnes propriétés. Entre autres, les projecteurs dans $\bar{H}$ et $\tilde{H}$ sont continus pour la topologie faible de H_0^1 et les fonctions de $\tilde{H}$ sont harmoniques dans Ω_+, c.-à-d. qu'elles satisfont l'équation $\Delta\tilde{u} = 0$ dans Ω_+. Si $u \in \mathcal{V}_\mu$, u se décompose de manière orthogonale en $\bar{u}+\tilde{u}$, où $\bar{u} \in \bar{H}$ et $\tilde{u} \in \tilde{H}$, mais il faut prendre garde au fait que les projections sortent de la variété. En accord avec l'idée intuitive que nous avons déjà mentionnée, lorsque μ est grand, la composante dans $\tilde{H}$ des fonctions de $\mathcal{V}_\mu$ d'énergie finie est presque nulle. Il semble donc naturel de comparer la géométrie de E contrainte à $\mathcal{V}_\mu$ à la géométrie de E sur $\bar{H} \cap \mathcal{V}_\mu$. Pour décrire cette dernière, introduisons le simplexe non linéaire

$$\mathcal{S} := \Big\{u = \sum_{i=1}^{n} s_i^{\frac{1}{\gamma+2}} \hat{u}_i \mid (s_1, \dots, s_n) \in \Delta\Big\} \subset \mathcal{V}_\mu,$$

où

$$\Delta := \Big\{(s_1, \dots, s_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^{n} s_i = 1\Big\}.$$

Pour $u = \sum_{i=1}^{n} s_i^{\frac{1}{\gamma+2}} \hat{u}_i \in \mathcal{S}$, notons

$$\varphi(s) := E(\sum_{i=1}^{n} s_i^{\frac{1}{\gamma+2}} \hat{u}_i) = \sum_{i=1}^{n} s_i^{\frac{2}{\gamma+2}} E(\hat{u}_i).$$

La géométrie de cette fonction $\varphi : \Delta \subset \mathbb{R}^n \to \mathbb{R}$ est un bon modèle "fini" de la géométrie de E sur $\mathcal{V}_\mu$ lorsque μ est grand.

Proposition 21. *Supposons que les hypothèses du Théorème* 20 *sont vérifiées. Considérons $\hat{u}_i$, $i = 1, \dots, n$, les minimants de E dans les variétés $\hat{\mathcal{V}}_i$, définies par* (53). *Les sommets* $(1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$

de Δ *sont des minimants locaux stricts de la fonction* $\varphi : \Delta \to \mathbb{R}$ *définie par*

$$\varphi(s) = \sum_{i=1}^{n} s_i^{\frac{2}{\gamma+2}} E(\hat{u}_i).$$

Si $P := \{i_1, \dots, i_k\}$, $2 \le k \le n$ *et*

$$\Delta_k := \{s = (s_1, \dots, s_k) \in \mathbb{R}_+^k \mid \sum_{j=1}^{k} s_j = 1\},$$

la fonction $\varphi_P : \Delta_k \to \mathbb{R}$ *définie par*

$$\varphi_P(s) := \sum_{j=1}^{k} s_j^{\frac{2}{\gamma+2}} E(\hat{u}_{i_j})$$

possède un unique maximum global, que nous notons c_P*, atteint en un point* $s^* = (s_1^*, \dots, s_k^*) \in int\,\Delta_k$*, c.-à-d. tel que pour tout* $j = 1, \dots, k$, $s_j^* > 0$*. De plus, si* $Q \subsetneq P$*, alors* $c_Q < c_P$*.*

Toutes ces valeurs particulières, minima locaux et maxima de φ restreints aux sous-simplexes, sont en fait des valeurs d'inf-max de φ pour des classes de minimax bien choisies. Ceci suggère que des valeurs d'inf-max de E dans $\mathcal{V}_\mu$ s'approchent de celles de E dans $\mathcal{S}$ lorsque $\mu \to \infty$. Plus précisément, définissons une projection convenable de $\mathcal{V}_\mu$ dans $\mathcal{S}$. Comme évoqué précédemment, si $u \in \mathcal{V}_\mu$, nous ne pouvons bien évidemment pas affirmer que $\bar{u} \in \mathcal{V}_\mu$. Cependant, n'importe quelle fonction u telle que $V_\mu(u) > 0$ possède un multiple dans $\mathcal{S}$. La fonction Q_μ définie sur $\operatorname{dom} Q_\mu = \{u \in H_0^1(\Omega) \mid V_\mu(u) > 0\}$ par

$$(Q_\mu u)(x) := [V_\mu(u)]^{-\frac{1}{\gamma+2}} u(x)$$

est donc un projecteur (continu) sur $\mathcal{V}_\mu$. Pour définir une bonne projection dans $\mathcal{S}$, nous tenons compte ensuite des contributions locales à la contrainte dans chaque sous-domaine ω_i. La somme pondérée

$$\sum_{i=1}^{n} [\hat{V}_i(u)]^{\frac{1}{\gamma+2}} \hat{u}_i,$$

où $\hat{V}_i(u) := \int_{\omega_i} a_+(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2}\, dx$, a par ailleurs l'avantage d'avoir une énergie inférieure à celle de la fonction initiale u, et si $u \in \mathcal{V}_\mu$, nous avons

$$\sum_{i=1}^{n} \hat{V}_i(u) = 1.$$

L'argument clé qui nous permet de comparer E sur $\mathcal{V}_\mu$ et $\mathcal{S}$ est le suivant. La projection $P_\mu : \mathcal{V}_\mu \to \mathcal{S}$ définie par

$$P_\mu u := \sum_{i=1}^{n} [\hat{V}_i(Q_\mu((\bar{u})_+)]^{\frac{1}{\gamma+2}} \hat{u}_i,$$

n'augmente presque pas l'action des fonctions de $\mathcal{V}_\mu$ d'énergie finie dès que μ est suffisamment grand. Plus précisément, nous démontrons dans [BGH] que pour tout $r > 0$ et $\delta > 0$, il existe $\mu_{r,\delta} > 0$ tel que

$$E(P_\mu u) - E(u) \leq \delta$$

pour tout $\mu \geq \mu_{r,\delta}$ et u vérifiant $E(u) \leq r$. Cette propriété très importante du projecteur P_μ permet de démontrer l'existence de niveaux inf-max de E dans $\mathcal{V}_\mu$ proche de ceux de E restreint à $\mathcal{S}$.

Les solutions 1-bosse sont proches (après projection) des minimants locaux de E dans $\mathcal{S}$. Nous les obtenons comme minimants locaux dans des sous-ensembles de fonctions dont les projections dans $\mathcal{S}$ sont proches d'un sommet. Ces sous-ensembles sont définis pour $i = 1, \dots, n$ par

$$\mathcal{F}_{i\mu} := \{u \in \mathcal{V}_\mu \mid \hat{V}_i(P_\mu u) = \hat{V}_i(R_\mu u) \geq \rho_i\},$$

où $\rho_i \geq \frac{2}{3}$, de sorte que ceux-ci soient disjoints. Il est clair que E atteint un minimant dans chaque sous-ensemble $\mathcal{F}_{i\mu}$. Un raisonnement par l'absurde montre que ces minimants appartiennent à l'intérieur des $\mathcal{F}_{i\mu}$. En effet, si ce n'est pas le cas, le minimant obtenu a une énergie plus grande que $\hat{u}_i \in \mathcal{F}_{i\mu}$, du moins pour μ assez grand. Chaque minimant est donc un point critique de E sous la contrainte. Lorsque $\mu \to \infty$, la projection dans $\mathcal{S}$ du minimant dans $\mathcal{F}_{i\mu}$ $(i = 1, \dots, n)$ converge vers une fonction 1-bosse à support inclus dans ω_i. Nous obtenons ainsi n familles différentes de solutions 1-bosse.

Pour obtenir les solutions multi-bosses, nous nous basons sur la structure d'inf-max de E sur $\mathcal{S}$. Fixons p fonctions $\hat{u}_i$ et supposons-les numérotées de 1 à p pour simplifier les notations. Notons $P = \{1, \dots, p\}$ l'ensemble d'indices correspondant. Construisons ensuite le simplexe non linéaire $\mathcal{S}_P$ sur les fonctions $\hat{u}_1, \dots, \hat{u}_p$,

$$\mathcal{S}_P := \Big\{u = \sum_{j=1}^{p} s_j^{\frac{1}{\gamma+2}} \hat{u}_j \mid (s_1, \dots, s_p) \in \Delta_p\Big\}.$$

Nous avons vu dans la Proposition 21 que la fonctionnelle E restreinte à $\mathcal{S}_P$ atteint un maximum global strict en un point intérieur w. Pour obtenir un point critique de E dont la projection est proche de w, il est naturel de considérer un principe de minimax. Définissons la classe H_P des fonctions $h \in C(\mathcal{S}_P, \mathcal{V}_\mu)$ telles que pour tout $u \in \partial\mathcal{S}_P$, $h(u) = u$ et pour tout $u \in \mathcal{S}_P$, $E(h(u)) \leq E(u)$. Celle-ci peut s'interpréter comme la classe des déformations continues de $Id_{\mathcal{S}_P}$ qui fixent

$$\partial \mathcal{S}_P := \{u = \sum_{j=1}^{p} s_j^{\frac{1}{\gamma+2}} \hat{u}_j \mid s_i = 0 \ \text{ pour un certain } \ i = 1, \ldots, p\}$$

et le long desquelles l'énergie décroît. Il est assez aisé de vérifier, en appliquant le théorème de *linking* (voir par exemple M. Willem [Wi]), que le niveau d'inf-max

$$\inf_{h \in H} \max_{u \in \mathfrak{S}_P} E(h(u))$$

est une valeur critique de E dans $\mathcal{V}_\mu$ si μ est suffisamment grand. Cependant, il n'est pas du tout évident que pour des ensembles d'indices P différents, cette caractérisation variationnelle mène à des points critiques différents et surtout que les solutions correspondantes soient des solutions p-bosses. En effet nous manquons d'informations concernant la localisation des points critiques. Pour contourner cette difficulté, nous délaissons cette caractérisation trop générale et basons notre approche sur des arguments de déformation. Nous localisons les déformations le long des lignes du flot gradient et identifions des régions du simplexe $\mathcal{S}$ dans lesquelles aucun point critique de E ne se projette lorsque μ est assez grand. D'autre part, nous nous assurons que les points critiques ainsi obtenus mènent bien à des familles de solutions p-bosses différentes.

En ce qui concerne la solution n-bosses, elle se déduit de la caractérisation variationnelle énoncée ci-dessus correspondant au choix d'indices $\{1, \ldots, n\}$. Dans ce cas, l'énergie du point critique est proche du maximum global de E sur $\mathcal{S}$, ce qui permet de le distinguer des autres. Pour obtenir une famille de solutions n-bosses il faut néanmoins recourir à une localisation des suites de Palais-Smale.

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- **Denis Bonheure, Patrick Habets et Luís Sanchez**, *Heteroclinics for fourth order symmetric bistable equations*, Atti Sem. Mat. Fis. Univ. Modena, à paraître.
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- **Denis Bonheure**, *Multitransition connections for fourth order equations with a bistable nonlinearity*, International Conference on Differential Equations (Equadiff 2003), Hasselt-Belgium July 22-26, World Scientific Publishing, à paraître.
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- **Denis Bonheure et Christian Fabry**, *Littlewoods problem for isochronous oscillators*, Institut de Mathématique Pure et Appliquée, Université Catholique de Louvain, Séminaire de mathématique no. 319, Janvier 2002, 10 pages.
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A Study of Heteroclinic Orbits for a Class of Fourth Order Ordinary Differential Equations

Contents

Introduction

In qualitative theory of differential equations, a prominent role is played by special classes of solutions, like periodic solutions or solutions to some kind of boundary value problems. When a system of ordinary differential equations has equilibria, i.e. constant solutions, whose stability properties are known, it is significant to study the connections between them by trajectories of solutions of the given system. These are called *homoclinic* or *heteroclinic* solutions, sometimes *pulses* or *kinks*, according to whether they describe a loop based at one single equilibrium or they "start" and "end" at two distinct equilibria.

These connections provide important informations on the dynamics of the system. For autonomous second order systems, for instance, homoclinics and heteroclinics separate, in the phase portrait, regions where solutions behave differently. When they appear as connections between saddles they prevent structural stability (see e.g. J. K. Hale and H. Koçak [**50**]). In addition to their relevance in the understanding of dynamical properties of the system, homoclinic and heteroclinic solutions appear in the context of a number of mathematical models for problems arising in Mechanics, Chemistry and Biology. For example, a classical model for studying diffusion in Biomathematics [**45, 56**] is provided by the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = \frac{\partial}{\partial x}\left(p(u)\frac{\partial u}{\partial x}\right) + g(u), \tag{1}$$

where H and p are positive functions and g is a positive function in $]0,1[$ such that $g(0) = g(1) = 0$. The first term on the right-hand side represents density dependent diffusion, while the term $\frac{\partial}{\partial x}H(u)$ accounts for convection effects. An important problem related to this equation is that of finding positive *travelling wave solutions*, i.e. waves that evolve at a constant speed, connecting the equilibria $u = 1$ to $u = 0$. This amounts to looking at the solutions of the second order ordinary differential equation

$$(p(u)u')' + (c - H'(u))u' + g(u) = 0, \tag{2}$$

satisfying the limit conditions

$$\lim_{x\to-\infty} u(x) = 1, \ \lim_{x\to+\infty} u(x) = 0.$$

In equation (2), the wave speed c is an unknown parameter. When $H(u)$ and $p(u)$ are constantly equal to 1 and $g(u)$ is the odd model nonlinearity $u - u^3$, the resulting equation (1) is usually referred to as the *Fisher-Kolmogorov equation*, sometimes also known as the *Allen-Cahn equation* or *Real Ginzburg-Landau equation.* When considering *standing waves*, i.e. stationary solutions, we are led to the simple model equation

$$-u'' + u^3 - u = 0, \tag{3}$$

which can be analyzed using elementary methods. Of course, as $u = 0$ and $u = 1$ are equilibrium states at different energy levels, a standing wave cannot connect these two equilibria. For standing waves, the relevant question is that of the existence of connections between the two stable states $u = -1$ and $u = +1$. This question is pertinent when studying physical systems which are *bi-stable.* The term bi-stable indicates that the corresponding evolution equation has two stable states $u(x) = \pm 1$ separated by a third one, which is unstable.

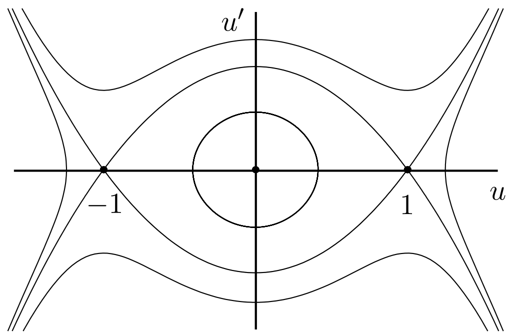


FIGURE 1. *The Fisher-Kolmogorov equation phase plane.*

For the stationary Fisher-Kolmogorov equation, a phase plane analysis, illustrated in Figure 1, shows that

$$u^{\pm}(x) = \pm \tanh\left(\frac{x+a}{\sqrt{2}}\right), \ a \in \mathbb{R},$$

are the only solutions connecting the stable states $u = \pm 1$, and these are monotone. The exact analytical form of these solutions is obtained by solving the first integral of energy.

In 1988, G. T. Dee and W. van Saarloos [**40**] introduced a higher-order model equation for bi-stable systems, adding a fourth order term in the evolution equation, which leads to the *Extended Fisher-Kolmogorov equation*

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \tag{4}$$

where γ is a real positive parameter. Here, even for the search of standing waves, no elementary method is available to deal with the existence of heteroclinic solutions.

An Overview

Methods. The study of homoclinic and heteroclinic solutions has a long history. Besides phase plane analysis, whose applicability is confined to second order autonomous equations, the study of such solutions has often been developed using the geometric theory of ordinary differential equations and dynamical systems techniques. In the classical approach, starting from an integrable system, we can analyse the dynamics of new systems that are small perturbations of the former. A simple example is provided by the addition of a small forcing term to an autonomous equation. In the nineteenth century, H. Poincaré [**83**] already studied perturbed time periodic systems. Many works have originated from Poincaré's results. In particular, Melnikov's theory provides instruments for the analysis of how homoclinics and heteroclinics are affected by perturbations on a Hamiltonian system. The main idea is that the existence of loops or connections at some rest points can be proved by analyzing the intersection properties of the stable and unstable manifolds through those equilibrium points. In the 1960's, V. K. Melnikov [**71**] proved by means of analytical methods the existence of homoclinics for non-conservative perturbations, leading to chaos. S. Smale [**104, 105**] then showed that in the presence of a transverse homoclinic point, the Poincaré map admits a Bernoulli shift structure. Similar ideas are present in works of G. D. Birkhoff [**11**]. We refer to J. Moser [**75**] and S. Wiggins [**117**] for further developments in these directions.

Starting mainly in the 1980's, a functional analytic approach added powerful tools to the research of connecting orbits for Hamiltonian systems. *Variational methods*, combining some classical ideas with modern *Critical Point Theory*, have thus provided a wealth of new results. A very large number of contributions were devoted to this topic, the main developments being due to A. Ambrosetti, S. V. Bolotin, V. Coti Zelati, I. Ekeland, P. H. Rabinowitz and E. Séré [**13, 87, 36, 37, 14, 6, 100, 88**]. The advantage of this approach comes from the fact that we can often bypass the question of transversality of the stable and unstable manifolds, whose verification is delicate in practice, or obtain weaker nondegeneracy conditions. Moreover, it leads to results of a global nature. Some comparisons with geometric methods and variational interpretations of Melnikov and Smale-Birkhoff theorems were inspected by A. Ambrosetti and M. Badiale [**4, 5**], and in [**73, 111**].

A third route to homoclinics and heteroclinics is provided by the Aubry-Mather Theory of monotone twist maps. This theory, which is also of a global nature, is well adapted to treat discrete dynamical systems. We refer the reader to V. Bangert [**8**] and J. Moser [**76**] for an introduction to this field and to J. N. Mather [**66**] for a more specific reference concerning existence results of connecting orbits. Although the Aubry-Mather Theory is based on variational principles and has common points with the approach used in this thesis, we do not discuss it at all.

The Model Equation. This thesis is devoted to the existence of heteroclinic solutions for a specific class of fourth order differential equations related to semilinear evolution equations of the form

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \tag{5}$$

where β is a real parameter. This equation serves as a model in studies of pattern formation in many physical, chemical or biological systems. It also arises in the theory of phase transitions near *Lifshitz points* [**119**]. We refer to [**82**] for further details.

When β is positive, it is related to the Extended Fisher-Kolmogorov equation (4). Indeed, by a scaling of the variable x, equation (5) can be written in the form (4), where the positive parameter γ is related to β by the formula $\beta = 1/\sqrt{\gamma}$.

When β is negative, equation (5) is related to the *Swift-Hohenberg equation*

$$\frac{\partial u}{\partial t} - \kappa u + (1 + \frac{\partial^2}{\partial x^2})^2 u + u^3 = 0, \tag{6}$$

where $\kappa \in \mathbb{R}$. This equation has been proposed by J. B. Swift and P. C. Hohenberg [**109**] as a model for analyzing Rayleigh-Bénard convection. When $\kappa > 1$, after rescaling, equation (6) can be written as

$$\frac{\partial u}{\partial t} + (\kappa - 1)^{3/2} \left(\frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} + u^3 - u \right) = 0$$

with $\beta = -2/\sqrt{\kappa - 1}$.

In many of the quoted problems, the main concern consists in analyzing the large-time behaviour of solutions of the evolution equations and the *attractor* of the corresponding dynamical system. As in various cases, the attractor consists of time-independent solutions, the set of bounded stationary solutions, such as periodic solutions, homoclinic or heteroclinic solutions, is of great interest. Also of interest is the existence of travelling waves. In the present work, we mainly focus on stationary heteroclinic solutions. These are time-independent solutions that spatially connect two uniform states. When looking at stationary solutions of (5), we are led to the autonomous equation

$$u'''' - \beta u'' + u^3 - u = 0. \tag{7}$$

Heteroclinics of (7) connecting -1 to $+1$ in the phase-space satisfy the following conditions

$$\lim_{x\to\pm\infty} (u, u', u'', u''')(x) = (\pm 1, 0, 0, 0). \tag{8}$$

Obviously, we can also consider connections from $+1$ to -1.

Nonlinear Schrödinger equations are also related to the model equation (7). When assuming harmonic spatial dependence ($v(x,t) = u(t)\mathrm{e}^{ikx}$ for some $k \in \mathbb{R}$), the solutions of the Schrödinger equation

$$i\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial^4 v}{\partial t^4} + |v|^2 v = 0$$

solve, after scaling, equation (7) with $\beta = 1/\sqrt{k}$. Then, a question of interest in nonlinear optic is the existence of pulse propagation. For equation (7), this amounts to looking for the existence of homoclinic solutions, i.e. solutions with property

$$\lim_{x\to\pm\infty} (u(x), u'(x), u''(x), u'''(x)) = (1, 0, 0, 0) \quad (\text{or } (-1, 0, 0, 0)). \tag{9}$$

Though we obtain some existence results for homoclinics, such solutions are not systematically studied in this thesis.

The set of bounded solutions of (7) has been the object of much research in the past ten years. In an impressive serie of papers [**78, 79, 80, 81**], L. A. Peletier and W. C. Troy performed a systematic study of periodic, homoclinic, heteroclinic and chaotic solutions of the model equation (7) for the parameter range $\beta \geq 0$. The parameter β plays a central role in the analysis of the behaviour of solutions of (7). Indeed, the nature of the equilibria ± 1 changes at two critical values $\beta = \pm\sqrt{8}$. The linearization of (7) at $u = \pm 1$ reads

$$v'''' - \beta v'' + 2v = 0, \tag{10}$$

where v stands respectively for $u - 1$ and $u + 1$. The eigenvalues of the associated characteristic equation

$$\lambda^4 - \beta\lambda^2 + 2 = 0$$

are

$$\lambda = \pm\sqrt{\frac{\beta \pm \sqrt{\beta^2 - 8}}{2}}. \tag{11}$$

When $\beta \geq \sqrt{8}$, the four eigenvalues are real so that $u = \pm 1$ are *saddle-nodes.* For $\beta \in (-\sqrt{8}, \sqrt{8})$, they are all complex with non-vanishing real parts. The equilibria are then called *saddle-foci.* When β passes below $-\sqrt{8}$, the eigenvalues become purely imaginary and therefore $u = \pm 1$ are *centers.* These different cases are depicted in Figure 2.

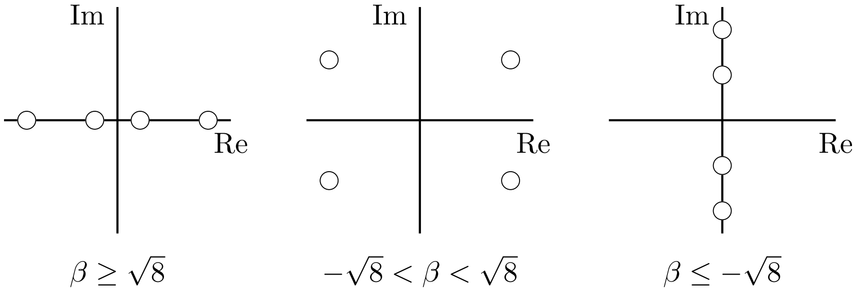


FIGURE 2. *The spectrum.*

The behaviour of the solutions of the linearizations of (7) around the equilibria provides important informations concerning the solutions of the nonlinear equation. Indeed, when none of the eigenvalues has a vanishing real part, it is well known that under some smoothness assumptions, the nonlinear flow and the flows defined by the linearizations are conjugate in neighbourhoods of the equilibria (see e.g. P. Hartman [**51**]). Consequently, when $\beta > -\sqrt{8}$, the solutions of (7) inherit some properties of the small solutions of (10) when they are close to $u = \pm 1$ (with small first, second and third derivatives). For example, we easily obtain a qualitative description of the shape of any heteroclinic at $\pm\infty$. Indeed, when ± 1 are saddle-nodes, the solutions of (10) that vanish at $+\infty$ or $-\infty$ are monotone, while in the saddle-foci case, they do oscillate around zero.

As previously mentioned, equation (7) can be written as

$$\frac{1}{\beta^2} u'''' - u'' + u^3 - u = 0 \tag{12}$$

after a suitable scaling of u. Hence, for large β, equation (12) can be treated as a fourth-order perturbation of the Fisher-Kolmogorov equation (3). We therefore expect that for large β the heteroclinic solutions of (7) are monotone. L. A. Peletier and W. C. Troy developed in [**78**] a topological shooting method especially adapted to track monotone heteroclinics. The main idea of a classical shooting method is to look at the way solutions change with respect to initial conditions (taken as parameters) at some fixed initial point. It turns out that their approach works well for $\beta \geq \sqrt{8}$. For this range of β, it is also remarkable that the uniqueness (up to translations and symmetry) of the heteroclinic solution of the Fisher-Kolmogorov equation extends to the fourth-order equation [**58, 114**]. It is worth pointing out the role of the assumption $\beta \geq \sqrt{8}$ in those results. Remember that as long as $\beta \geq \sqrt{8}$, the roots of the characteristic equation associated to the linearization of (7) around the equilibria are real. The squares of these roots are

$$\tau_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - 8}}{2}.$$

Now observe that the linear operator $D^4 - \beta D^2 + 2I$ can be factorized as $(D^2 - \tau_+ I)(D^2 - \tau_- I)$. Letting $v = 1 - u$, equation (7) written in terms of v yields

$$v'''' - \beta v'' + 2v = v^2(3 - v). \tag{13}$$

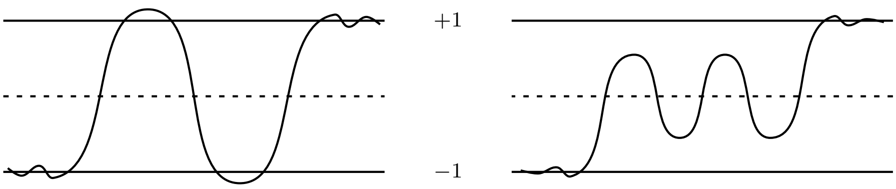
Thus, the solutions (v, w) of the system of second order equations

$$\begin{cases} v'' - \tau_+ v = w, \\ w'' - \tau_- w = v^2(3 - v), \end{cases} \tag{14}$$

lead to solutions of (7). This formulation as a system turns out to be very powerful to obtain estimates of v (and therefore of u). These estimates rely on a repeated application of the *Strong Maximum Principle* [**84**], which can be used once we know the sign on the right-hand side in each equation of the system.

When $0 < \beta < \sqrt{8}$, the preceding formulation as a system and the subsequent use of the Maximum Principle are no longer at hand. Anyway, a monotone heteroclinic cannot exist in this parameter range. Indeed, as already emphasized, the linearization around the equilibria display oscillatory solutions so that any heteroclinic solution of (7) oscillates around ± 1 in its tail, i.e. when $x \to \pm\infty$. This oscillatory behaviour of the solutions close to the equilibria makes a shooting method much more tedious. One of the greatest difficulties when dealing with oscillatory solution graphs is to control the convergence at infinity. However, L. A. Peletier and W. C. Troy adapted their arguments in [**79**]. They singled out two families of odd heteroclinics thanks to a careful analysis. The two families differ by the amplitude of the oscillations. The first family consists of so-called *multi-transition* solutions as all the successive local extrema between the zeros are located outside the region $[-1, 1]$, see Figure 3 (a). It contains, for each $n \in \mathbb{N}$, a solution whose profile displays $2n + 1$ jumps from -1 to $+1$ and two oscillatory tails around -1 and $+1$. In the second family, the amplitude of the oscillations is smaller than 1 so that the corresponding solutions are *single-transition* heteroclinics, see Figure 3 (b). The solutions in this family may also be classified according to their number of oscillations around 0.

The dynamics of equation (7) with $\beta < 0$ is much less understood than the one of the Extended Fisher-Kolmogorov equation. Numerical experiments [**112**] suggest that a large variety of those solutions found



(a) (b)

FIGURE 3. *(a) The profile of a multi-transition heteroclinic with three jumps. (b) The profile of a single-transition heteroclinic with five zeros.*

for β positive still exist for a certain range of negative values of β. However, the limitation of the shooting method of L. A. Peletier and W. C. Troy was pointed out by J. B. van den Berg [**113**].

The Variational Approach. In this thesis, we are more particularly concerned with the variational structure of equation (7). It is an elementary fact that solutions of (7) correspond to critical points of the *action functional*

$$\mathcal{L}(u) := \int_I L_\beta(u(x), u'(x), u''(x))\, dx, \tag{15}$$

where L_β is the second order *Lagrangian*

$$L_\beta(u, u', u'') := \frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2 - 1)^2. \tag{16}$$

The last term of the Lagrangian is often called the *potential*, although according to the usual terminology in classical mechanics,

$$V(u) = -\frac{1}{4}(u^2 - 1)^2$$

should be called the potential. Solutions of (7) are critical points of $\mathcal{L}$ in different functional spaces depending on the type of solution considered. Also, the integral in (15) is taken on various sets I according to the domain of the solution and the functional space on which $\mathcal{L}$ is defined.

As heteroclinics are defined on $\mathbb{R}$, we consider $I = \mathbb{R}$ in (15) leading to the functional

$$\mathcal{F}_\beta(u) := \int_{\mathbb{R}} \frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2 - 1)^2\, dx. \tag{17}$$

This functional is well defined for functions u having first and second square integrable derivatives and being such that the potential is integrable. Taking into account conditions (8), which are satisfied by heteroclinics, we can define $\mathcal{F}_\beta$ in the space

$$\{u : \mathbb{R} \to \mathbb{R} \mid u + 1 \in H^2(\mathbb{R}^-),\ u - 1 \in H^2(\mathbb{R}^+)\}.$$

Indeed, it is well known (see e.g. H. Brezis [**20**]) that if $u+1 \in H^1(\mathbb{R}^-)$, then

$$\lim_{x\to-\infty} u(x) = -1$$

and u is bounded in every compact interval $[-T, 0]$. A similar observation can be made on $\mathbb{R}^+$ so that we infer

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{4}(u^2-1)^2\,dx &= \int_{\mathbb{R}^-} \frac{1}{4}(u^2-1)^2\,dx + \int_{\mathbb{R}^+} \frac{1}{4}(u^2-1)^2\,dx \\ &\leq \frac{C}{4}\left(\|u+1\|^2_{L^2(\mathbb{R}^-)} + \|u-1\|^2_{L^2(\mathbb{R}^+)}\right)\end{aligned}$$

for some positive constant C and therefore $\mathcal{F}_\beta(u)$ is finite.

The existence of heteroclinic solutions of (7) via variational arguments was investigated for the first time by L. A. Peletier, W. C. Troy and R. C. A. M. VanderVorst [**77**] and W. D. Kalies, R. C. A. M. VanderVorst [**55**]. For $\beta \geq 0$, $\mathcal{F}_\beta$ is a positive functional. It is therefore natural to look for heteroclinics as minimizers of the action functional. L. A. Peletier et al. [**77**] proved the existence of a minimizer of $\mathcal{F}_\beta$ in the subset of odd functions of the space

$$\mathcal{E} := \{u : \mathbb{R} \to \mathbb{R} \mid u(0) = 0,\ u+1 \in H^2(\mathbb{R}^-),\ u-1 \in H^2(\mathbb{R}^+)\}. \tag{18}$$

This space can be alternatively defined by

$$\{u + \chi \mid u \in H^2(\mathbb{R}),\, u(0) + \chi(0) = 0\},$$

where for example χ is a C^∞ function such that

$$\chi(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ +1 & \text{if } x \geq +1 \end{cases}$$

and $-1 \leq \chi(x) \leq +1$ for $-1 \leq x \leq +1$. It was shown in W. D. Kalies et al. [**55**] that actually all the minimizers in $\mathcal{E}$ are odd. To obtain an odd heteroclinic solution, it is sufficient to look for critical points of the functional

$$\mathcal{F}^+_\beta(u) := \int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2-1)^2\right) dx \tag{19}$$

in the space

$$\mathcal{E}^+ := \{u \mid u - 1 \in H^2(\mathbb{R}^+),\ u(0) = 0\}. \tag{20}$$

Indeed, the condition $u''(0) = 0$ is a natural boundary condition fulfilled by any critical point of $\mathcal{F}^+_\beta$ in $\mathcal{E}^+$ so that their odd extensions solve the Euler-Lagrange equation (7) on $\mathbb{R}$. Also, it is easy to check that conditions (8) at $\pm\infty$ are fulfilled.

These arguments extend easily to a functional defined from a second order positive Lagrangian with a symmetric potential having two non-degenerate minima (i.e. with non-vanishing second derivative) at the same energy level and superquadratic grows at $\pm\infty$.

When $0 \leq \beta < \sqrt{8}$, as we already mentioned, infinitely many heteroclinic solutions of (7) appear. For this range of the parameter β, the equilibrium solutions ± 1 are saddle-foci and it is known that heteroclinic connections between such kind of equilibria can exhibit a complex structure, see [**21, 26, 41**] and the references therein. W. D. Kalies and R. C. A. M. VanderVorst [**55**] constructed for (7) so called *multi-bump* solutions, i.e solutions with multiple oscillations separated by large distances. The usual methods used to obtain such solutions are rather tricky and require either a careful study of the stable and unstable manifolds or a certain kind of nondegeneracy condition on a primary connection whose well separated copies are then glued together [**26, 37, 100**]. W. D. Kalies, J. Kwapisz and R. C. A. M. VanderVorst [**54**] introduced a direct method to find multi-transition solutions. A typical example of the shape of such solutions is depicted in Figure 4. We recall that the term multi-transition refers to the fact that the graph of such solutions consists of multiple jumps from one equilibrium to the other. One jump is then called a transition. Also, such solutions are qualitatively different from multi-bump solutions, as the distance between two successive transitions is not necessarily large. The method of W. D. Kalies et al. consists in minimizing the action functional $\mathcal{F}_\beta$ in specific subspaces of functions having a common homotopy type. Basically, the homotopy type describes the trajectory of any function in the uu'-plane by recording the number of transitions from one equilibrium to the other and counting the number of turns it makes around -1 and $+1$ in between the transitions. Their method perfectly handles oscillatory graphs and is therefore efficient when $0 \leq \beta < \sqrt{8}$.

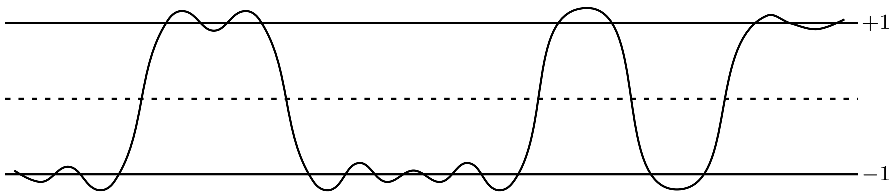


FIGURE 4. *The profile of a typical multi-transition heteroclinic with oscillations between the jumps.*

General Framework. Many arguments of the above quoted papers used in the minimization process, both in the whole space and in homotopy classes, rely on the positivity of the Lagrangian $L_\beta(u, u', u'')$ and therefore on the positivity of the parameter β. First attempts to consider sign changing Lagrangians were made by P. Habets, L. Sanchez, M. Tarallo and S. Terracini [**49**] and D. Bonheure, L. Sanchez, M. Tarallo and S. Terracini [**18**]. These authors considered the more general functional

$$\mathcal{F}_g(u) := \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx, \tag{21}$$

whose Euler-Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{22}$$

Here f is assumed to be a double-well potential, not necessarily symmetric, with bottoms at ± 1 at the same energy level and g is neither necessarily constant nor positive. The interest of considering a non-constant function g can be checked in [**48**]. The main idea of [**18, 49**] is to impose a condition on g to ensure a lower bound on the action $\mathcal{F}_g(u)$ over admissible functions. The following assumption does the job: g is such that for some function $\tilde{g} \in C(\mathbb{R})$, some $k < 1$ and all $u \in \mathbb{R}$,

$$g(u) \geq \tilde{g}(u), \ |\tilde{G}(u)| \leq k\sqrt{8f(u)}, \tag{23}$$

where

$$\tilde{G}(u) := \int_0^u \tilde{g}(s)\, ds.$$

This condition can be seen as a good balance between the possible negativity of g and the positivity of the potential. Some weak regularity assumptions have to be made on f and g to deal with the functional $\mathcal{F}_g$, but it should be stressed that the approach in [**18, 49**] does not require the nondegeneracy of the equilibria ± 1 as implicitly assumed when considering the model potential in $\mathcal{F}_\beta$. This rather weak assumption on the potential does not allow to define the functional $\mathcal{F}_g$ in an affine translate of $H^2(\mathbb{R})$ so that a slight modification of the functional setting has to be performed. Namely, it is sufficient to search for a minimizer in the functional space

$$\{u \in C^1(\mathbb{R}) \mid u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}) \text{ and } \lim_{x \to \pm\infty} u(x) = \pm 1\}.$$

Though assumption (23) allows g to take negative values, it rules out a negative constant. Indeed, $\tilde{g}$ must satisfy

$$\int_{-1}^0 \tilde{g}(s)\, ds = \int_0^{+1} \tilde{g}(s)\, ds = 0.$$

However, it was observed in V. J. Mizel, L. A. Peletier and W. C. Troy [**72**] that for $\beta < 0$ with $|\beta|$ small, the functional $\mathcal{F}_\beta$ is still bounded from below. For instance, D. Bonheure, P. Habets and L. Sanchez [**16**] showed that there exists $\beta_0 \in]-0.9481\ldots, 0[$ such that the functional $\mathcal{F}_\beta$ has a finite infimum in $\mathcal{E}$ for all $\beta \geq \beta_0$, while for each $\beta < \beta_0$, $\mathcal{F}_\beta$ is unbounded from below. Furthermore, the critical parameter β_0 is characterized by

$$\beta_0 = \inf\{\beta < 0 \mid \inf_{\mathcal{E}} \mathcal{F}_\beta \geq 0\}. \tag{24}$$

A sharp estimate of β_0 is still missing. In D. Bonheure et al. [**16**], we considered the functional $\mathcal{F}_g$, assuming that g takes negative values everywhere but we only dealt with symmetric functionals with potentials having nondegenerate minima. Also, the functional is assumed to be bounded from below. This assumption holds true when the negative part of g, i.e. $g^-(u) := \max(0, -g(u))$, is small.

When $u = \pm 1$ are saddle-foci, i.e. when

$$g(\pm 1)^2 < 4f''(1),$$

we expect the existence of infinitely many heteroclinic solutions of (22). As underlined above, the arguments of W. D. Kalies et al. do crucially depend on the positivity of the Lagrangian so that their results do only cover equation (22) when g is non-negative. In [**15**], we partially extended those results to even Lagrangians that can take either signs. To deal with such Lagrangians, only an a priori lower bound on the action along admissible functions is required. The latter holds for example under the compatibility assumption (23) or when g^- is small. Multi-transition connections for equation (22) can then be obtained as local minimizers in classes of functions with prescribed profiles.

Outline of the Thesis

Minimization Methods and Second Order Systems. In Chapter 1, we give a brief general introduction to the variational methods and we present the *direct methods of the Calculus of Variations* that are used for minimizing a functional. We then give a first insight of how minimizing arguments can be used to track heteroclinic solutions. With this aim in view, we review some of the basic results concerning the search of heteroclinic solutions for second order *Hamiltonian systems.* We consider systems like

$$u'' - \nabla V(u) = 0, \tag{25}$$

where $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is a potential (according to the usual classical terminology in mechanics, $-V$ should be called the potential) with several equilibria at the same (minimum) level of the energy. Roughly speaking, the common underlying idea is the following: suppose that V has two minima, say ξ and η; we look for heteroclinic connections between ξ and η as minimizers of the action functional

$$\int_{\mathbb{R}} \left(\frac{|u'(t)|^2}{2} + V(u(t)) \right) dt \tag{26}$$

in an appropriate class of functions u defined in $\mathbb{R}$ so that $u(-\infty) = \xi$ and $u(+\infty) = \eta$. We start with a very simple scalar autonomous equation having two equilibrium points to highlight the main features of the minimizing process. We then proceed successively to non-autonomous equations, conservative reversible Hamiltonian systems of multiple pendulum type and periodic Hamiltonian systems. When the potential has more than two equilibrium points, we cannot ensure the existence of a heteroclinic solution between all pair of minima. However, given two distinct rest points ξ, η, we can obtain, under simple assumptions, a *heteroclinic chain* connecting ξ and η, i.e. a finite set of heteroclinics $\{v_1, \dots, v_j\}$ such that $v_0(-\infty) = \xi$, $v_{i+1}(-\infty) = v_i(+\infty)$ for $i = 0, \dots, j-1$ and $v_j(+\infty) = \eta$. The results of this chapter are mainly due to P. H. Rabinowitz and T. O. Maxwel. Further references are given throughout the chapter.

Minimization of Positive Functionals. From Chapter 2, we really get to the heart of the matter. We first focus on the model functional $\mathcal{F}_\beta$ defined by (17). In this case, the minimizing process is quite simple as we can take benefit of the symmetry of the potential to work with minimizing sequences of odd functions. Another great advantage of the Lagrangian $L_\beta(u, u', u'')$ is that the potential $f(u) = \frac{1}{4}(u^2-1)^2$ is superquadratic at $\pm\infty$ and has nondegenerate minima, i.e. $f''(\pm 1) \neq 0$. This property implies that for u close to $+1$ (respectively -1), $f(u)$ behaves like the square of the L^2-norm of $u-1$ (respectively $u+1$). This last feature provides a simple argument to prove the weak convergence of a subsequence of the minimizing sequence. In Section 2.1, we prove the following theorem.

THEOREM 1. *For all $\beta \geq 0$, the functional $\mathcal{F}_\beta : \mathcal{E} \to \mathbb{R}$ defined by* (17) *and* (18) *has a global minimizer, which is a heteroclinic solution of* (7) *connecting -1 to $+1$. Furthermore, any minimizer is odd and positive in $]0, +\infty[$.*

The original proof of the existence statement of Theorem 1 was carried out in L. A. Peletier et al. [**77**]. We give here a proof which is closer to W. D. Kalies and R. C. A. M. VanderVorst [**55**].

In Section 2.2, we consider the more general functional $\mathcal{F}_g$ defined by (21), assuming that g is a non-negative function. Here, we deal with a larger class of potentials which are not necessarily even. Namely, we only require the potential to be bounded from above by parabolas $\alpha(u+1)^2$ and $\beta(u-1)^2$, close to, respectively -1 and $+1$. The proof that $\mathcal{F}_g$ achieves a minimum is more tedious and requires a careful analysis of the properties of the minimizing sequence to ensure that the weak limit converges to ± 1 at $\pm\infty$. As we already mentioned, the rather weak assumption on the potential does not allow to define the functional $\mathcal{F}_g$ in the space $\mathcal{E}$. We then search for a minimizer in the functional space

$$\mathcal{H} := \{u \in C^1(\mathbb{R}) \mid u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}),\ \lim_{x \to \pm\infty} u(x) = \pm 1\}. \quad (27)$$

The key argument is that given two neighborhoods N_{-1} of $(-1, 0)$ and N_{+1} of $(+1, 0)$ in the (u, u')-plane, the time it takes for a function of $\mathcal{H}$ to travel from N_{-1} to N_{+1} is bounded from above in terms of its action. On the other hand, choosing a sufficiently small neighborhood of $(+1, 0)$, the functions that have limit $+1$ at $+\infty$ and start from this neighborhood stay close to $+1$ in the future. A similar observation can be made concerning the convergence to -1 in the past. These properties ensure a sufficient control on the minimizing sequence. Combined with a priori estimates, these arguments allow to extract a weak converging subsequence that has its limit in the right space. We now state the main result of the section.

THEOREM 2. *Let $g \in C^1(\mathbb{R})$ be a non-negative function. Assume that $f \in C^1(\mathbb{R})$ is a double well non-negative potential such that*
(a) $f(u) = 0$ *if and only if* $u = \pm 1$*,*
(b) *for some* $0 < a < 1$ *and* $\alpha > 0$*,*

$$\frac{f(u)}{(u-1)^2} \le \alpha, \textit{ for } |u-1| < a,$$
$$\frac{f(u)}{(u+1)^2} \le \alpha, \textit{ for } |u+1| < a,$$

(c) $\liminf\limits_{|u|\to+\infty} f(u) > 0$.

Then the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R}$ defined by (21) *and* (27) *has a minimizer u, which is a solution of* (22) *satisfying*

$$\lim_{x\to\pm\infty} (u(x), u'(x)) = (\pm 1, 0).$$

Observe that we obtain a weaker result than in Theorem 1 as we cannot conclude in general that u is a "true" heteroclinic solution satisfying conditions (8). The proof we present follows the arguments that we introduced in D. Bonheure et al. [**18**].

We then analyse, in Section 2.3, the qualitative properties of the minimizers. Here a striking difference appears with the arguments presented in Section 1.2.2. Indeed, when minimizing a functional in a certain space, we often want to be able to modify locally any function by another one which is in the same space, has better properties and has a lower action. When dealing with a second order equation and its associated functional, we usually only have to worry about keeping functions continuous. As our functionals are defined in subspaces of $H^2_{\mathrm{loc}}(\mathbb{R})$, the problem is more delicate. For instance, any modification has to keep the functions at least C^1. When considering, for example, the Fisher-Kolmogorov equation

$$-u'' + u^3 - u = 0$$

and the associated functional

$$\int_{\mathbb{R}} \left(\frac{u'^2}{2} + \frac{1}{4}(u^2-1)^2 \right) dx$$

defined in the space

$$\{u \in H^1_{\mathrm{loc}}(\mathbb{R}) \mid u(0) = 0 \text{ and } \lim_{x\to\pm\infty} u(x) = \pm 1\},$$

it is easily seen that the minimizer takes its values in $[-1, 1]$ and is monotone increasing. Indeed, if u is a minimizer which does not fulfil these conditions, then

$$v(x) = \max(-1, \min(u(x), 1))$$

has a lower action and if v is not monotone, we can convert it into a monotone function with modifications that decrease its action. These modifications, which keep functions in $H^1_{\mathrm{loc}}(\mathbb{R})$, do not necessarily produce functions of class C^1 and therefore, in general, the modified functions do not belong to $H^2_{\mathrm{loc}}(\mathbb{R})$. As already pointed out, it is not true in general that minimizers of $\mathcal{F}_\beta$ in $\mathcal{E}$ are monotone.

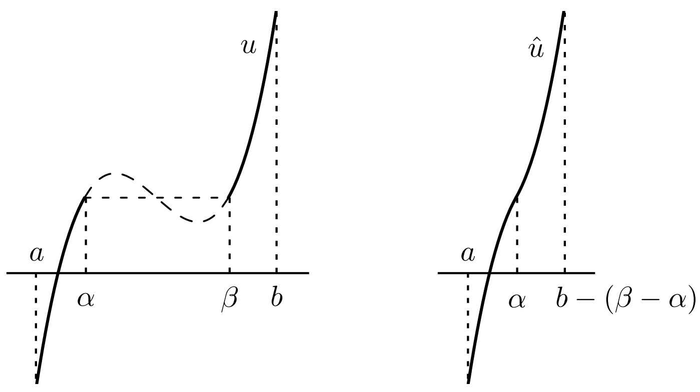
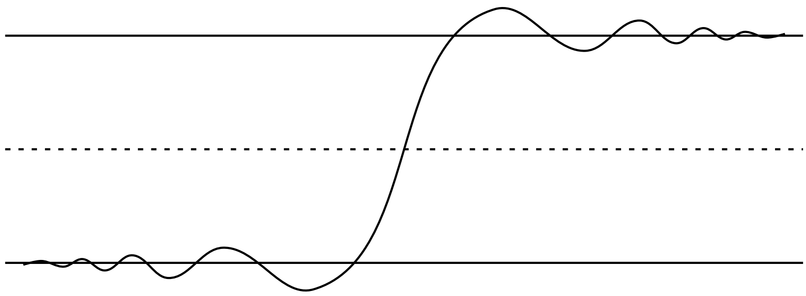


FIGURE 5. *The clipping process.*

In Section 2.3.1, we investigate the admissible cutoffs in $H^2_{\mathrm{loc}}(\mathbb{R})$. We recall the clipping process introduced by W. D. Kalies et al. [**54**] and a generalized version of D. Bonheure and L. Sanchez [**17**]. Basically, if $u \in H^2[a, b]$, $u(\alpha) = u(\beta)$ and $u'(\alpha) = u'(\beta)$ for some $\alpha < \beta$ in $[a, b]$, we say that the interval $[\alpha, \beta]$ can be *clipped out*, meaning that we can define a H^2 function $\hat{u}$ on the interval $[a, b-(\beta-\alpha)]$ which coincides with u on the interval $[a, \alpha]$ and with the $\beta - \alpha$ translate of $u_{|[\beta,b]}$ on the interval $[\alpha, b-(\beta-\alpha)]$. We say that $\hat{u}$ is a *clip* of u. A clipping is illustrated in Figure 5. It is straightforward that any clip of u has a lower action at least as long as the Lagrangian is positive.

Using this tool, we can prove that any minimizer of both $\mathcal{F}_\beta$ and $\mathcal{F}_g$ is monotone in its transitions. This means none of the local extrema can be reached in the region $[-1,1]$ between the first visit of $+1$ and the last visit of -1. In the symmetric case, i.e. for $\mathcal{F}_\beta$ or $\mathcal{F}_g$ with f and g even, more characteristics can be established as any minimizer has to be odd. This allows to prove that a minimizer only vanishes once. Then, if ± 1 are saddle-foci, the profile of the minimizer consists in one monotone transition from -1 to $+1$ and two oscillatory tails, one around $+1$ as $x \to +\infty$ and the symmetric one around -1 as $x \to -\infty$, see Figure 6. Furthermore, in the tail around $+1$, the successive local maxima of the minimizer decrease to $+1$, while the successive local minima increase to $+1$.



$+1$

-1

FIGURE 6. *The profile of the so-called* principal heteroclinic, *i.e. the heteroclinic with the smallest action, in the symmetric saddle-foci case.*

Sign Changing Lagrangians. Chapter 3 is devoted to sign changing Lagrangians, namely Lagrangians of the form

$$L_g(u, u', u'') := \frac{1}{2}(u''^2 + g(u)u'^2) + f(u), \tag{28}$$

where the function $g \in C^1(\mathbb{R})$ changes sign and f is a double-well potential. In Section 3.1, we prove the following result.

THEOREM 3. *Let $f \in C^1(\mathbb{R})$ be a double well non-negative potential satisfying the assumptions of Theorem* 2. *Assume that $g \in C^1(\mathbb{R})$ is such that for some function $\tilde{g} \in C(\mathbb{R})$, some $k < 1$ and all $u \in \mathbb{R}$, condition* (23) *holds. Then the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R}$ defined by* (21) *and* (27) *has a minimizer u, which is a solution of* (22) *satisfying*

$$\lim_{x \to \pm\infty} (u(x), u'(x)) = (\pm 1, 0).$$

Many of the arguments used to prove Theorem 2 extend to Theorem 3. However, some steps require special attention as, for example, the weak lower semi-continuity of the functional is not obvious once g

changes sign. First of all, we need to prove that $\mathcal{F}_g$ is bounded from below in $\mathcal{H}$. The proof of Theorem 3 has been previously presented in P. Habets et al. [**49**] under the additional assumption that g is non-negative close to ± 1. This hypothesis, which was basically used to prove the weak lower semi-continuity of the functional, turns out to be unnecessary. Also some of the arguments used in the original proof are simplified in a similar way as done in our survey D. Bonheure and L. Sanchez [**17**].

As we already asserted, the compatibility condition (23) rules out negative constant function g. We deal with strictly negative functions g in Section 3.2. Here, more restrictive assumptions are needed. The main result of the section goes as follows.

THEOREM 4. *Let $f \in C^1(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ be even functions such that $f(1) = 0$ and for some $k > 0$, $\beta < \sqrt{8k}$ and all $u \geq 0$,*

$$f(u) \geq k(u-1)^2 \quad \text{and} \quad g(u) \geq -\beta.$$

Assume further that

$$\inf_{\mathcal{E}} \mathcal{F}_g > -\infty,$$

where $\mathcal{E}$ and $\mathcal{F}_g$ are defined from (18) *and* (21). *Then the functional $\mathcal{F}_g : \mathcal{E} \to \mathbb{R}$ has a minimizer, which is a heteroclinic solution of* (22) *connecting -1 to $+1$ and having exactly one zero. Moreover, any minimizer is odd.*

The hypothesis that $\mathcal{F}_g$ is bounded from below in $\mathcal{E}$ may seem very restrictive. However, as previously mentioned, this is weaker than considering non-negative Lagrangians. For example, given f satisfying the first part of the assumptions of Theorem 4, we can always find $\beta(f) \in (0, \sqrt{8k})$ such that $\mathcal{F}_g$ is bounded from below if for all $u \in \mathbb{R}$, $g(u) \geq -\beta$ with $\beta \leq \beta(f)$. It is also worth mentioning that although Theorem 4 covers some cases that do not enter into the framework of Theorem 3, it requires an explicit lower bound on g, which Theorem 3 does not. In this sense, these theorems are complementary. We formerly disclosed Theorem 4 in D. Bonheure et al. [**16**].

In Section 3.3, we partially extend Theorem 4 to non-symmetric functionals. Namely, we prove the following result.

THEOREM 5. *Let $f \in C^2(\mathbb{R})$ be a non-negative double well potential satisfying the assumptions of Theorem 2. Assume, moreover, that*
(a) *± 1 are nondegenerate minima of f;*
(b) *$g \in C^2(\mathbb{R})$ is such that $g(\pm 1)^2 < 4f''(\pm 1)$;*
(c) *there exist $\varepsilon > 0$ and $\tilde{g} \in C(\mathbb{R})$ such that $g(u) - \tilde{g}(u) \geq \varepsilon$ and*

$$\inf_{\mathcal{E}} \mathcal{F}_{\tilde{g}} > -\infty,$$

where $\mathcal{E}$ *is defined by* (18) *and* $\mathcal{F}_{\tilde{g}}$ *is defined according to* (21). *Then the functional* $\mathcal{F}_g : \mathcal{E} \to \mathbb{R}$ *has a minimizer, which is a heteroclinic solution of* (22) *starting from* -1 *and ending at* $+1$.

Half of assumption (b) $(-2\sqrt{f''(\pm 1)} < g(\pm 1))$ is necessary to reach the conclusion, while the positive parameter ε in assumption (c) only seems to be of a "technical" assistance. Using condition (c), we easily derive an a priori bound for the L^2-norm of u' and u'' and another one for the integral of the potential. Assumption (b), which means the equilibria ± 1 are saddle-foci, then controls the time a quasi-minimizer may spend close to the minima. Theorem 5 has not previously been published.

Multi-transition Heteroclinics. In Chapter 4, we consider the question of multiplicity of heteroclinic solutions when the minima of the potential are saddle-focus equilibria. When projected in the configuration plane (u, u'), a heteroclinic orbit yields a curve connecting the points $(\pm 1, 0)$. Choosing two oriented loops e_1 and e_2 around respectively $(-1, 0)$ and $(+1, 0)$, see Figure 7, a homotopy type can be associated to any curve connecting $(\pm 1, 0)$. This homotopy type records the number of transitions between these points and the oscillations around them between the transitions. Hence, every orbit connecting $(-1, 0)$ to $(+1, 0)$ can be represented by a word of the form

$$e_1^{\theta_{2m}} \cdot e_2^{\theta_{2m-1}} \cdot \ldots \cdot e_1^{\theta_2} \cdot e_2^{\theta_1},$$

where $\theta(u) = (\theta_1, \ldots, \theta_{2m}) \in \mathbb{N}^{2m}$. Observe that the word starts with the first visit at $(+1, 0)$ and stops recording after the last visit around $(-1, 0)$. Moreover, the components of the vector $\omega(u) = 2\theta(u)$ contain the exact number of crossings that u makes with either -1 or $+1$ between two successive transitions. The vector $\omega = 0$ is therefore associated to functions with a single transition.

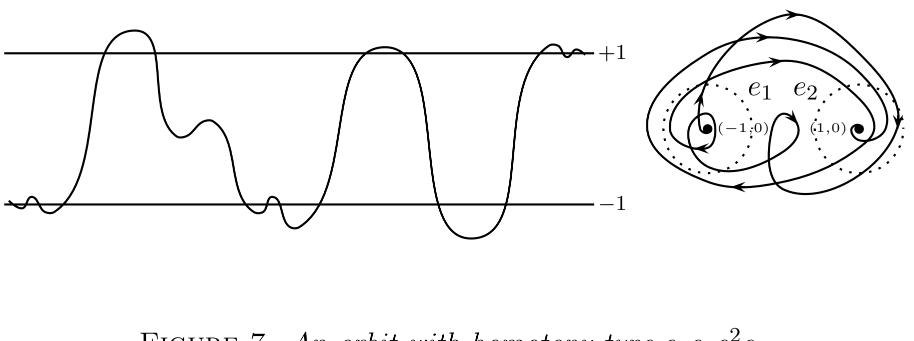


FIGURE 7. *An orbit with homotopy type* $e_1 e_2 e_1^2 e_2$.

For each vector $\omega \in 2\mathbb{N}^{2m} \cup \{0\}$, we may define the homotopy class $M(\omega)$ consisting of functions of $\mathcal{E}$ having winding vector $\omega/2$. For $\beta \geq 0$, the functional $\mathcal{F}_\beta$ is positive so that we can define

$$c_\omega := \inf_{u \in M(\omega)} \mathcal{F}_\beta(u). \tag{29}$$

If the infimum is achieved by a function in the interior of $M(\omega)$, then a local minimizer of $\mathcal{F}_\beta$ with the corresponding properties solves the Euler-Lagrange equation (7). The main difficulty is to show that minimizing sequences in $M(\omega)$ have weak limits in the interior of the class. Indeed, minimizing sequences can approach the boundary of $M(\omega)$ so that the limit function could gain or lose complexity. For example, tangential crossings of ± 1 could appear due to a coalescence of two or more crossings or to a new spurious oscillation around one equilibrium. The oscillatory behaviour of solutions in a neighborhood of a saddle-focus equilibrium plays then a crucial role to control the minimizing sequences. Efficient tools were developed by W. D. Kalies et al. to adjust functions of the boundary of $M(\omega)$. Basically these tools rely on a cut and paste technique which allows to delete spurious oscillations or replace pieces of functions tangent to ± 1 with pieces of small oscillating orbits around $(\pm 1, 0, 0, 0)$ in the phase-space. Another source of trouble in the minimization process comes from a possible lack of compactness when passing to a weak limit. Indeed, the distance between two transitions or two crossings of either -1 or $+1$ could grow to infinity. Here again, the oscillatory properties of the orbits, close to a saddle-focus, prevent losses of complexity at the limit.

The arguments of W. D. Kalies et al. extend to the functional $\mathcal{F}_g$ assuming that $f \in C^2(\mathbb{R})$ has exactly two nondegenerate global minima at $u = \pm 1$ and grows superquadratically at $\pm\infty$. However, they do crucially depend on the positivity of the Lagrangian and therefore on the sign of g. When looking at larger classes than those defined by the homotopy types, we can relax the assumptions on g. Namely, we define for each $p \geq 0$, the subset $\mathcal{E}_p^+ \subset \mathcal{E}^+$ (defined by (20)) consisting of functions whose odd extension to $\mathbb{R}$ make $2p+1$ transitions. More precisely, a function $u \in \mathcal{E}^+$ belongs to the subclass $\mathcal{E}_p^+$ if there exist $0 = x_0 < x_1 < \ldots < x_p < x_{p+1} = +\infty$ such that

$$u(x)(-1)^{i+p} > 0 \text{ for } x \in \ (x_i, x_{i+1})$$

and

$$\max_{(x_i, x_{i+1})} u(x)(-1)^{i+p} > 1.$$

An element of $\mathcal{E}_3^+$ is depicted in Figure 8. We then prove under conve-

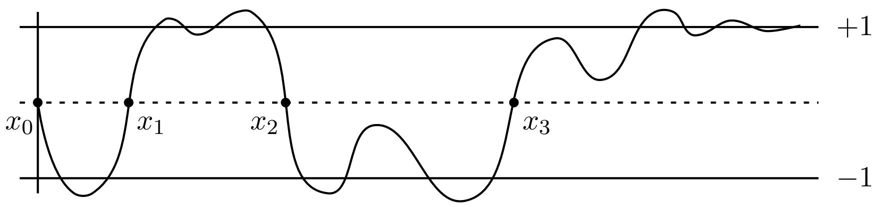


FIGURE 8. *A function in* $\mathcal{E}_3^+$.

nient assumptions, that the functional $\mathcal{F}_g^+ : \mathcal{E}^+ \to \mathbb{R}$ defined by

$$\mathcal{F}_g^+(u) := \int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \tag{30}$$

has a local minimum in each of these subspaces. The odd extension of these minima are multi-transition heteroclinics of (22). In comparison with the result of W. D. Kalies et al., the minimima in $\mathcal{E}_p^+$ correspond to heteroclinic solutions of homotopy type $\omega = (\omega_1, \ldots, \omega_{2p})$ with $\omega_i = 2$ for every $i = 1, \ldots, 2p$. Our approach requires the symmetry of the functional in an essential way, though we think this assumption is probably unnecessary. In Section 4.2.1, we prove the following theorem.

THEOREM 6. *Let* f, $g \in C^2(\mathbb{R})$ *satisfy the hypotheses of Theorem* 4. *Assume moreover* ± 1 *are nondegenerate minima and* $g(\pm 1)^2 < 4f''(\pm 1)$. *Then, for every* $p \in \mathbb{N}$, $\mathcal{F}_g^+$ *has, in each subspace* $\mathcal{E}_p^+$, *a local minimizer* u_p *whose odd extension to* $\mathbb{R}$ *is a heteroclinic solution of* (22) *connecting* -1 *to* $+1$ *and having exactly* $2p + 1$ *zeros.*

The framework of Theorem 3 is considered in Section 4.2.2.

THEOREM 7. *Let* f, $g \in C^2(\mathbb{R})$ *be even functions satisfying the hypotheses of Theorem* 3. *Assume moreover* ± 1 *are nondegenerate minima of* f *and* $g(\pm 1)^2 < 4f''(\pm 1)$. *Then the conclusion of Theorem 6 holds.*

The proof of Theorem 6 and a sketch of that of Theorem 7 were originally published in D. Bonheure [**15**]. Complementary details were given in our survey D. Bonheure and L. Sanchez [**17**]. Equivalent results can be obtained for homoclinic solutions. We show in Section 4.3 how to adapt the definition of the classes $\mathcal{E}_p^+$ to find even homoclinic connections. Of course, these classes have to be defined in another functional space that takes conditions (9) into account.

Connections between Non-consecutive Equilibria. In the closing chapter of this thesis, we discuss the case of an equation similar to (22), where f is a triple well potential and we address the following question. Does the dynamics possess a heteroclinic orbit connecting the extremal equilibria?

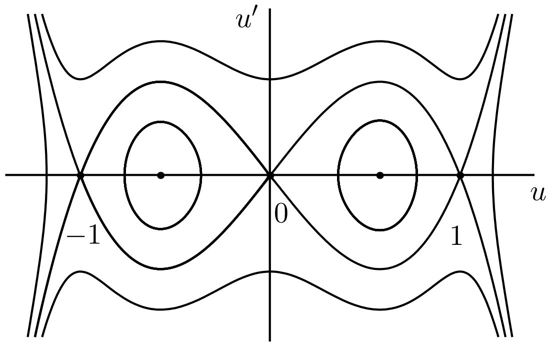


FIGURE 9. *The three stable-states system phase plane.*

Let us take a look at a classical mechanics analogy. Consider a moving particle in a potential characterized by three hills of equal height. To fix the ideas, suppose the tops of the peaks are located at -1, 0 and $+1$ and consider a motion starting from -1 at time $t \to -\infty$. As the potential energy is identical at the top of each hill, the law of energy conservation implies that the particle needs an infinite amount of time to reach the top of the second hill and therefore cannot pass through the middle-equilibrium. The corresponding phase portrait is depicted in Figure 9.

For the fourth order equation (22), the energy reads

$$E(u) = u'''u' - \frac{1}{2}u''^2 + \frac{g(u)}{2}u'^2 + f(u).$$

For the corresponding equation of motion, a particle does not need to come at rest at the top of the second hill as the constant of motion can be satisfied with a non-zero u' due to the presence of the new terms $u'''u' - \frac{1}{2}u''^2$. In other words, the intersection of the zero energy manifold and the space $u = 0$ is not reduce to a point. We answer the above question positively at least when the middle-equilibrium is of saddle-focus type.

THEOREM 8. *Assume that $f \in C^1(\mathbb{R})$ is a triple well non-negative potential such that*
(a) $f(u) = 0$ *if and only if* $u \in \{-1, 0, 1\}$,
(b) f *is of class* C^2 *in a neighbourhood of* 0 *and* $f''(0) \neq 0$,
(c) *for some* $0 < a < 1/2$ *and* $\alpha > 0$,

$$\frac{f(u)}{(u-1)^2} \leq \alpha, \text{ for } |u-1| < a,$$
$$\frac{f(u)}{(u+1)^2} \leq \alpha, \text{ for } |u+1| < a,$$

(d) $\liminf_{|u|\to+\infty} f(u) > 0$.

Assume also that $g \in C^2(\mathbb{R})$ is such that $g(0)^2 < 4f''(0)$ and for some function $\tilde{g} \in C(\mathbb{R})$, some $k < 1$ and all $u \in \mathbb{R}$, condition (23) *holds. Then the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R}$ defined by* (21) *and* (27) *has a minimizer u, which is a solution of* (22) *satisfying*

$$\lim_{x \to \pm\infty} (u(x), u'(x)) = (\pm 1, 0).$$

We mention that this question is relevant in the study of Ginzburg-Landau models of amphiphilic systems [**48**]. We refer to Section 5.1 for a brief description of these models. Theorem 8 first appeared in D. Bonheure et al. [**18**]. Functionals with a multiple well potential were earlier considered in W. D. Kalies et al. [**54**] but under fairly stronger assumptions.

Open Questions

Many problems related to those in this thesis remain open. The most important one consists in understanding the geometry of the functional $\mathcal{F}_\beta$ in $\mathcal{E}$ when $\beta < \beta_0$, that is when it becomes unbounded from below. Another challenge could be to analyse, for $0 \le \beta < \sqrt{8}$, the variational nature of the family of single-transition solutions found by L. A. Peletier and W. C. Troy thanks to their shooting method. We suspect that those solutions correspond to minimax critical levels between the local minima that lead to the multi-transition solutions. As far as we know, no results in this direction have been obtained yet. Other open problems are mentioned throughout the text.

About the Presentation

Each chapter ends with a *Notes and Comments* section. These include the bibliographical notes concerning the chapter, some remarks and complementary results that we have chosen not to consider in detail. We also give references for some extensions of results presented here in a simplified version, we briefly discuss problems for which less is known, we refer to open questions or we simply mention some related topics that did not find their way in this monograph.

Chapter 1

The Variational Methods and Heteroclinics for Second Order Equations and Systems

Though the main topic of the thesis concerns fourth order differential equations, we focus in this chapter on the existence of heteroclinic solutions for second order differential equations and systems. We have chosen to consider these problems because, to our opinion, they provide a nice introduction to the use of variational arguments in the study of heteroclinic orbits.

In the first section of the chapter, we briefly discuss the history of the variational methods and we introduce the direct methods of the Calculus of Variations as well as the application to the search of heteroclinic connections for second order differential equations.

We then survey basic results concerning scalar second order differential equations and Hamiltonian systems of second order equations. The results we present are often elementary versions of known results mainly due to P. H. Rabinowitz (see [**92**] for a comprehensive overview). Some of these could suggest further research in the context of either scalar or systems of fourth order ordinary differential equations.

1.1. The Variational Methods

The study of variational principles has a long history and a lot of famous names are associated to it. Though the ancient Greeks already considered isoperimetric problems and optimal principles in optic, the first modern contribution is attributed to P. Fermat who postulated that light propagates following a path of least possible time and therefore derived the laws of refraction. Fermat's works inspired the brothers Johann and Jakob Bernoulli who are considered among the founders of the *Calculus of Variations*. In 1696, Johann Bernoulli proposed in *Acta Eruditorum*, as a challenge to other mathematicians, the *brachistochrone problem* which asks what shape must have a wire for a bead to slide (under the action of gravity alone) from one end to the other in the shortest possible time. Five solutions were obtained: I. Newton, Jakob Bernoulli, G. W. Leibniz and G. de L'Hôpital in addition to Johann Bernoulli. The last cited was not the first to consider the brachistochrone problem. Galileo in 1638 had studied the problem in his famous work *Discourse on two new sciences*. However, Galileo was wrong in his analysis, claiming the path of quickest descent would be an arc of a circle. L. Euler published in 1744 the first textbook [**44**] on the Calculus of Variations. In an appendix, L. Euler declares *"every effect in nature follows a maximum or minimum rule"*. The same belief is present in P. L. Maupertuis' work on the widely known *least action principle*.

Using the modern terminology, we say that a given boundary value problem has a variational structure if the solutions of the latter can be obtained as critical points of an associated functional defined in a suitable space of functions. Maybe the most celebrated variational problem is the so-called Dirichlet boundary value problem for the Laplace equation on a domain $\Omega \subset \mathbb{R}^N$

$$\begin{aligned} \Delta u(x) &= 0, & x &\in \Omega \\ u(x) &= u_0(x), & x &\in \partial\Omega, \end{aligned} \tag{1.1}$$

whose variational formulation consists in finding critical points of the energy functional

$$u \to \int_\Omega |\nabla u(x)|^2 \, dx,$$

in the space

$$\{u \in H^1(\Omega) \mid u(x) = u_0(x) \text{ for all } x \in \partial\Omega\}.$$

Gauss admitted without proof that the above energy functional, usually called *the Dirichlet integral*, has a minimum over all sufficiently regular functions whose restriction on the boundary of a bounded domain Ω is fixed. His deep conviction is known as the *Dirichlet Principle*. The use of the variational methods quickly spread to many problems of mechanics or physics such as determining the distribution of an electrical charge on the surface of a conductor that minimizes the energy of the associated electrical field, see e.g. L. Dirichlet [**42**] or C. F. Gauss [**46**]. All these applications were treated without rigor using sometimes confused models. The common assumption was that the evidence in nature of the well-posedness of the problem is a sufficient condition for the corresponding mathematical problem to have a solution. K. Weierstrass was one of the first to criticize the Dirichlet Principle as he found an example of a functional that do not achieve its minimum. His example

[**116**] consists in minimizing the functional

$$I : u \to \int_{-1}^{1} |tu'(t)|^2 \, dt$$

over all continuously differentiable functions $u : [-1, 1] \to \mathbb{R}$ subjected to the boundary conditions $u(\pm 1) = \pm 1$. Using the family of functions

$$u_\varepsilon(t) = \frac{\arctan(\frac{t}{\varepsilon})}{\arctan(\frac{1}{\varepsilon})},$$

for $\varepsilon > 0$, he showed that the infimum of I over this set of functions is 0 though the value 0 is clearly not achieved.

However, at the beginning of the 20th century, D. Hilbert [**52**] closely followed by H. Lebesgue [**60**], tidied up the earlier works on the Dirichlet problem and gave a neat proof of the Dirichlet Principle. From then, many actors, including K. Weierstrass, C. Arzela, M. Fréchet, D. Hilbert and H. Lebesgue, contributed to secure the bases of the Calculus of Variations. They were followed by L. Tonelli who was the main protagonist in the abstract formulation of the so-called direct methods of the Calculus of Variations. Since then, these methods have been extensively used and generalized, making the most of the developments in functional analysis and in particular of the theory of convex sets and reflexive Banach spaces. With the *Ljusternik-Schnirelman theory* [**65**], a *global analysis* was born as not only minimizers but all possible critical points of variational integrals were considered. Another route towards a global theory was initiated by M. Morse [**74**]. These two approaches were the outburst of many contributions in Critical Point Theory developed by either topologists and analysts, the most well-known among them being J. Milnor, R. Palais, S. Smale, E. Rothe, F. H. Clark, A. Ambrosetti, I. Ekeland and P. H. Rabinowitz. In the late 1970's, P. H. Rabinowitz revolutionized the study of Hamiltonian systems by his use of the variational methods to prove the existence of periodic solutions. The understanding of the set of periodic solutions of Hamiltonian systems was already a preoccupation of H. Poincaré who used a least action principle to study the closed orbits of a conservative system with two degrees of freedom. However, a comprehensive variational approach to periodic orbits of Hamiltonian systems took a long time to arise. In fact, the question whether any given energy surface carries a periodic solution has only recently been answered. For a long time it was considered hopeless to approach the existence problem for periodic solutions via the action functional. The main obstacle is that the functional is unbounded both from above and from below. This makes the use of the direct methods of the Calculus of Variation powerless (as it only deals with absolute minima). Surprisingly, P. H. Rabinowitz [**85**] was able to overcome this difficulty using extensions of the global analysis of the

Ljusternik-Schnirelman theory [**65**] and Morse theory [**74**]. Later, other contributions by F. H. Clarke and I. Ekeland [**34, 35**] provided a "dual" variational principle, which allows again to look for a solution as a minimizer of an appropriate variational problem.

At the end of the 1980's, still considering Hamiltonian systems, P. H. Rabinowitz confronted his route to periodic solutions to the search of connecting orbits between equilibrium points. The advantage when considering heteroclinic orbit between different rest points is that the functional is bounded from below, which is not the case when looking for a loop to a single equilibrium point. However, in both cases, we have to face an additional difficulty inherent to the nature of those solutions. Namely, heteroclinic and homoclinic solutions share an unbounded domain: $\mathbb{R}$.

Let us now focus on the direct methods of the Calculus of Variations and consider the minimization problem

$$\min\{I(u) \mid u \in X\},$$

where X is a space of functions and $I : X \to \mathbb{R} \cup \{+\infty\}$ is a functional bounded from below. As in the finite dimensional case, one way to study this problem is to search for the zeros of the derivative of I. We then try to solve the equation

$$I'(u) = 0$$

known as the *Euler-Lagrange equation* (which own its name to W. Hamilton) and study the second variation of I around the solutions. This approach is usually called *classical* and was extensively developed by J. L. Lagrange, B. Riemann, K. Weierstrass, C. G. Jacobi, W. Hamilton and others. The direct methods consist in dealing directly with the functional I. To explain the main ideas, let us start with a finite dimensional case. A simple way to prove the existence of a minima is to find a *minimizing sequence*, i.e. a sequence

$$(u_n)_n \subset X \text{ such that } \lim_{n\to+\infty} I(u_n) = \inf_X I,$$

which belong to a closed bounded set. Once we have such a sequence, we can extract a convergent subsequence, which necessarily converges to a minimum if I is continuous. For instance, this approach works fine if I is only *lower semi-continuous*, i.e. such that

$$\liminf_{k\to+\infty} I(u_k) \geq I(u) \text{ for every sequence } (u_k)_k \subset X \text{ such that } u_k \to u,$$

as we then conclude that

$$\inf_X I = \lim_{n\to+\infty} I(u_n) = \liminf_{n\to+\infty} I(u_n) \geq I(u),$$

which clearly implies that the equality holds. If we want to extend this analysis to the infinite dimensional case, then we immediately encounter a difficulty since a closed bounded case is in general not compact. In fact, such a set is often compact for a weaker topology than the usual one. As a consequence, the functional I should be lower semi-continuous with respect to the weak topology which ensures the convergence of a subsequence. More precisely, let X be a normed space and denote by X^* its dual. We recall that $(u_n)_n \subset X$ *converges weakly* to $u \in X$ if and only if

$$\langle u_n, u^* \rangle \to \langle u, u^* \rangle$$

for every $u^* \in X^*$. This convergence is usually denoted by

$$u_n \rightharpoonup u.$$

The nice property of the weak topology is that uniformly bounded sequences are precompact once X is reflexive. Therefore to ensure the compactness of the minimizing sequence in the weak topology, we only require a uniform bound on the sequence. Summing up, the direct methods require to work with a reasonable normed space X, a *weakly lower semi-continuous* functional I, i.e. such that

$$\liminf_{k \to +\infty} I(u_k) \geq I(u) \text{ for every sequence } (u_k)_k \subset X \text{ such that } u_k \rightharpoonup u$$

and to find a bounded minimizing sequence (for the norm of X). If X is a reflexive Banach space, then we can conclude under the previous assumptions that I achieves a minimum in X.

Let us now discuss the application of the direct methods to the search of heteroclinic solution. Consider a simple second order equation

$$u'' - a(t)F'(u) = 0,$$

where $F : \mathbb{R} \to \mathbb{R}$ is a smooth non-negative function that vanishes at some points ξ, η and a is a positive function, bounded away from zero. In order to find solutions that connect ξ to η, we minimize the functional

$$u \to \int_{\mathbb{R}} \left(\frac{u'^2}{2} + a(t)F(u) \right) dt,$$

in a reasonable space of functions that take both conditions

$$\lim_{t \to -\infty} u(t) = \xi \quad \text{and} \quad \lim_{t \to +\infty} u(t) = \eta$$

into account. These last conditions prevent from defining the functional in a Banach space. We could possibly work within an affine translate of a Banach space like $H^1(\mathbb{R}) + \chi$, where χ is a function connecting ξ to η such as

$$\chi(x) := \begin{cases} \xi & \text{if } x \leq 0, \\ t\eta + (1-t)\xi & \text{if } 0 \leq x \leq 1, \\ \eta & \text{if } x \geq 1. \end{cases}$$

However, this presumes $u-\xi \in L^2(\mathbb{R}^-)$ and $u-\eta \in L^2(\mathbb{R}^+)$ though the natural integrability condition fulfilled by a minimizer is

$$\int_{\mathbb{R}} F(u)\,dx < +\infty.$$

Without particular assumptions on the function F, the latter does not imply that a function u connecting η to ξ belong to $H^1(\mathbb{R}) + \chi$. It is therefore more convenient to define the action functional in the space

$$\Gamma(\xi,\eta) := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}) \mid u(-\infty) = \xi,\ u(+\infty) = \eta\}$$

and to adapt the usual direct methods to our needs. As we cannot endow $\Gamma(\xi,\eta)$ with a norm, we build a "homemade" minimizing process using the informations that are available on a minimizing sequence. First, it is easily seen that a minimizing sequence $(u_n)_n \subset \Gamma(\xi,\eta)$ is such that

$$\sup_{n\in\mathbb{N}} \|u_n'\|_{L^2(\mathbb{R})} < +\infty.$$

On the other hand, under reasonable assumptions on the functions a and F, we can prove that the minimizing sequence is uniformly bounded for the L^∞-norm. We may therefore combine Ascoli-Arzela Theorem and the compactness of the sequence $(u_n')_n$ for the weak topology in $L^2(\mathbb{R})$ to deduce the existence of a subsequence that we still denote by $(u_n)_n$ for simplicity, such that for some $u \in H^1_{\mathrm{loc}}(\mathbb{R})$,

$$u_n \to u$$

uniformly on compact intervals and

$$u_n' \rightharpoonup u'$$

in $L^2(\mathbb{R})$. This makes u a nice candidate for a minimizer. However, u does not necessarily belong to our functional space. Indeed, u may have been attracted by other equilibria than ξ and η at respectively $-\infty$ and $+\infty$. On the other hand, we also have to check that the functional is lower semi-continuous with respect to this "convergence" of the subsequence.

In the next sections, these arguments are confronted to various situations. We first consider an elementary scalar autonomous equation to point out the main arguments that are further applied to more evolved situations. Many frameworks have been considered in the literature and a lot of properties of heteroclinic solutions have been studied. It is hopeless to give a complete panorama on all existing results so that we have selected very basic situations referring therefore to the original papers for further refinements.

We close this brief introduction to the variational methods and the direct methods of the Calculus of Variations with some general references to these topics. H. H. Goldstine [**47**] describe in a very pleasant book the evolution of the Calculus of Variations. For comprehensive textbooks on the methods, we recommend I. Ekeland [**43**], B. Dacorogna [**39**], J. Mawhin and M. Willem [**67**], L. Sanchez [**99**], M. Struwe [**108**], P. H. Rabinowitz [**86**], M. Ramos [**96**] and M. Willem [**118**]. The recent survey of P. H. Rabinowitz [**92**] is a good starting point for who is interested in the application of the variational methods to Hamiltonian systems. Finally, let us mention the contributions of A. Ambrosetti and M. Badiale [**4, 5**] who investigated the relationship between variational and geometric methods usually used to study connecting orbits of Hamiltonian systems. Though those approaches seem apparently different in nature, they showed that an adequate application of variational arguments allow to treat the classical Poincaré-Melnikov perturbation results.

1.2. Basic Arguments for Scalar Equations

1.2.1. Phase Plane Analysis. Recall that a good description of the various kinds of solutions of the mathematical pendulum equation

$$u'' + a\sin u = 0, \tag{1.2}$$

($a > 0$), is provided by the representation of the corresponding trajectories in the phase plane (u, u'). These are level curves of the energy, that is, they are a locus of points (u, u') of the form

$$\frac{u'^2}{2} + a(1 - \cos u) = k \tag{1.3}$$

for some constant $k \in \mathbb{R}^+$. It is well known that the energy function

$$E(u, u') := \frac{u'^2}{2} + a(1 - \cos u)$$

is constant along solutions of (1.2).

When $k = 0$ in (1.3), we obtain the stable equilibria $u = 2n\pi$, where $n \in \mathbb{Z}$. If $0 < k < 2a$, the trajectories are closed curves corresponding to periodic solutions. For $k > 2a$, we obtain unbounded trajectories corresponding to solutions with periodic derivative. Finally, $k = 2a$ in (1.3) yields a locus consisting of the unstable equilibria $u = (2n + 1)\pi$, where $n \in \mathbb{Z}$ and the graphs of the functions

$$u' = \pm\sqrt{2a(1 + \cos u)}, \quad u \neq (2n + 1)\pi \ \text{ for all } n \in \mathbb{Z}. \tag{1.4}$$

For instance, the solutions having as trajectories the graphs of these functions in $]-\pi, \pi[$ have the property that

$$u(+\infty) := \lim_{t\to+\infty} u(t) = \pi,$$

$$u(-\infty) := \lim_{t\to-\infty} u(t) = -\pi$$

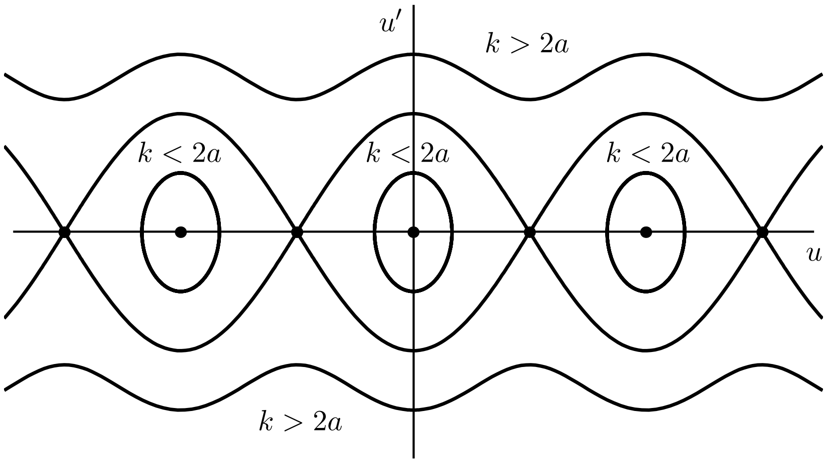


FIGURE 1.1. *The pendulum phase plane.*

and

$$\lim_{t\to\pm\infty} u'(t) = 0$$

or the same conditions with the roles of $+\infty$ and $-\infty$ reversed. Hence these trajectories connect two distinct (consecutive) unstable equilibria. They are called *heteroclinics* and any underlying solution is called a *heteroclinic solution* of (1.2). In this example it is apparent that they separate regions of the (u, u')-plane where the solutions of (1.2) have a different nature, see Figure 1.1.

Given the physical meaning of (1.2), it is sometimes preferable to depict trajectories not in a plane but in a cylinder (which is a plane where the points (u, u') and (v, v') are identified if and only if $u \equiv v \pmod{2\pi}$ and $u' = v'$). Then $(-\pi, 0)$ and $(\pi, 0)$ are in fact the same equilibrium and a trajectory connecting these points in the plane becomes a trajectory with equal limits at $\pm\infty$. We would then rather speak of a *homoclinic*. However, if one forgets the 2π-periodicity of the potential $a(1 - \cos u)$, or, for that matter, if one modifies it outside, say, the interval $]-\pi, \pi[$, the consideration of heteroclinics is meaningful.

1.2.2. The Autonomous Case. Let us consider a more general autonomous scalar equation

$$u'' - f(u) = 0, \tag{1.5}$$

where $f \in C([-1, 1])$ is a function such that

(A1) $f(\pm 1) = 0$;

(A2) there exists a primitive F of f such that $F(-1) = F(1) = 0$ and $F(u) > 0$ for all $u \in]-1, 1[$.

Hence, equation (1.5) has two equilibria, $u = \pm 1$, at the (same) zero level of the potential. As for solutions of (1.5) energy is conserved, that is

$$\frac{u'^2}{2} - F(u) = K \tag{1.6}$$

for some constant K, it makes sense to look for heteroclinic solutions connecting -1 and 1, i.e. solutions such that

$$u(\pm\infty) := \lim_{t\to\pm\infty} u(t) = \pm 1$$

and

$$u'(\pm\infty) := \lim_{t\to\pm\infty} u'(t) = 0$$

or the same properties with the roles of the $+\infty$ and $-\infty$ reversed. In fact, for such solutions we must have $K = 0$ in (1.6) and the corresponding phase plane trajectories are given explicitly by

$$u' = \pm\sqrt{2F(u)}, \qquad -1 < u < 1.$$

These are equations with separable variables. With the $+$ sign, for example, integrating we obtain

$$\int_0^u \frac{dv}{\sqrt{2F(v)}} = t + C, \tag{1.7}$$

for some constant C. In particular, the equilibria ± 1 are not reached in finite time whenever

$$\int_0^{\pm 1} \frac{dv}{\sqrt{F(v)}}$$

diverges. This condition is obviously satisfied provided that f is locally Lipschitz. More generally, it also holds if F is at most quadratic near its minima, that is if there exists $c > 0$ so that

$$F(u) \le c(u \pm 1)^2$$

in a neighborhood of -1 and $+1$ respectively.

Now we look for a heteroclinic of (1.5) from another angle. Instead of using elementary integration techniques, we show that such a solution can be characterized by a variational property. Needless to say, it may seem cumbersome to treat that simple problem in such an involved way, but since the variational method has an important role to play in the search of heteroclinics for systems and non-autonomous equations, it is worth to grasp the main ideas in an uncomplicated case.

Formally, equation (1.5) is the Euler-Lagrange equation of the functional

$$\mathcal{I}(u) := \int_{\mathbb{R}} \left(\frac{u'^2}{2} + F(u) \right) dt, \tag{1.8}$$

where $F(u)$ is the primitive of f given in assumption (A2). We look for heteroclinics of (1.5) as minimizers of $\mathcal{I}$ in the functional space

$$\left\{u \in H^1_{\mathrm{loc}}(\mathbb{R}, [-1,1]) \mid \ u(\pm\infty) = \pm 1\right\}.$$

Actually, it will be clear that we may consider the space

$$\Gamma := \left\{u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \mid \ u(\pm\infty) = \pm 1\right\} \tag{1.9}$$

and confine ourselves to functions taking values in $[-1,1]$ by simply extending F by 0 on $]-\infty,-1[\cup]+1,+\infty[$, which we assume from now on.

PROPOSITION 1.1. *Assume $f \in C([-1,1])$ satisfies assumptions* (A1) *and* (A2)*. If u minimizes the functional $\mathcal{I} : \Gamma \to \mathbb{R} \cup \{+\infty\}$ defined by* (1.8) *and* (1.9)*, then u is a classical solution of* (1.5) *satisfying $u'(\pm\infty) = 0$.*

PROOF. For any function $\varphi \in C^1_c(\mathbb{R})$, for all $\tau \in \mathbb{R}$, $u + \tau\varphi \in \Gamma$ and $\mathcal{I}(u+\tau\varphi)$ is differentiable as a real function of the parameter τ. Since u minimizes $\mathcal{I}$ in Γ, the function $\mathcal{I}(u+\tau\varphi)$ achieves a minimum at $\tau = 0$. We then compute

$$0 = \left.\frac{d}{d\tau}\right|_{\tau=0} \mathcal{I}(u+\tau\varphi) = \int_{\mathbb{R}} (u'\varphi' + f(u)\varphi)\, dt$$

and, by the Du Bois-Reymond Lemma, u satisfies (1.5).

Finally, since u satisfies (1.6) for some $K \in \mathbb{R}$ and there exist sequences $t_n \to \pm\infty$ with $u'(t_n) \to 0$, we infer that $K = 0$ and we then conclude that $u'(\pm\infty) = 0$. □

THEOREM 1.2. *If $f \in C([-1,1])$ satisfies assumptions* (A1) *and* (A2)*, the functional $\mathcal{I} : \Gamma \to \mathbb{R} \cup \{+\infty\}$ defined by* (1.8) *attains a minimum. A minimizer $u \in \Gamma$ is a heteroclinic solution of* (1.5) *connecting -1 to $+1$ and satisfying $-1 \le u(t) \le 1$ for all $t \in \mathbb{R}$.*

PROOF. Let $(u_n)_n \subset \Gamma$ be a minimizing sequence that is $u_n \in \Gamma$ for every $n \in \mathbb{N}$ and $\mathcal{I}(u_n) \to \inf_\Gamma \mathcal{I}$.

Step 1 - Modification of the minimizing sequence. By defining

$$v_n = \sup(-1, \inf(u_n, 1))$$

and observing that $\mathcal{I}(v_n) \le \mathcal{I}(u_n)$, we may assume without loss of generality that $-1 \le u_n \le 1$.

For each $\varepsilon > 0$, we find an interval $[s_n, t_n]$ such that $u_n(s_n) = -1+\varepsilon$, $u_n(t_n) = 1-\varepsilon$ and

$$-1+\varepsilon \le u_n(t) \le 1-\varepsilon \quad \text{for all } t \in [s_n, t_n].$$

If u_n takes values greater than $-1+\varepsilon$ in $]-\infty, s_n]$, we may choose $s_n' < s_n$ so that $u_n(s_n') = -1+\varepsilon$ and $-1 \le u_n \le -1+\varepsilon$ in $]-\infty, s_n']$. In the same way we find $t'_n \geq t_n$ so that $u_n(t'_n) = 1 - \varepsilon$ and $1 - \varepsilon \leq u_n \leq 1$ in $[t'_n, +\infty[$. We then define the new function

$$U_n(t) = \begin{cases} u_n(t - s_n + s'_n), & \text{if } t \leq s_n, \\ u_n(t), & \text{if } s_n \leq t \leq t_n, \\ u_n(t + t'_n - t_n), & \text{if } t \geq t_n. \end{cases}$$

By the translation invariance of the integrals and the fact that the integrand is positive it is clear that

$$\mathcal{I}(U_n) \leq \mathcal{I}(u_n)$$

so that $(U_n)_n \subset \Gamma$ is again a minimizing sequence, which in addition satisfies $-1 \leq U_n \leq -1 + \varepsilon$ in $]-\infty, s_n]$ and $1 - \varepsilon \leq U_n \leq 1$ in $[t_n, +\infty[$. Notice that the sequence $t_n - s_n$ is bounded since

$$\mathcal{I}(U_n) \geq (\min_{-1+\varepsilon \leq z \leq 1-\varepsilon} F(z))(s_n - t_n).$$

Observe also that as $\mathcal{I}$ is translation invariant, we may assume that $s_n = 0$.

Step 2 - Convergence. Now, since $\sup_n \|U_n\|_{L^\infty} \leq 1$ and $\sup_n \|U'_n\|_{L^2}$ is bounded, we apply the diagonal procedure to extract a subsequence that we still denote by $(U_n)_n$, and we obtain a function $u \in H^1_{\text{loc}}(\mathbb{R})$ such that

$$U_n \overset{C_{\text{loc}}(\mathbb{R})}{\longrightarrow} u,$$

i.e. uniformly in compact intervals and

$$U'_n \overset{L^2(\mathbb{R})}{\rightharpoonup} u'.$$

In addition, we have $t_n \to \bar{t} > 0$.

Combining the weak lower semi-continuity of the L^2-norm and Fatou's lemma we infer

$$\begin{aligned} \mathcal{I}(u) &\leq \liminf_{n\to+\infty} \int_{\mathbb{R}} \frac{{U'_n}^2}{2}\, dt + \liminf_{n\to+\infty} \int_{\mathbb{R}} F(U_n)\, dt \\ &\leq \lim_{n\to+\infty} \mathcal{I}(U_n) = \inf_\Gamma \mathcal{I}. \end{aligned} \tag{1.10}$$

Step 3 - $u \in \Gamma$. By uniform convergence, we have $-1 \leq u(t) \leq -1 + \varepsilon$ for $t \leq 0$ and $1 - \varepsilon \leq u(t) \leq 1$ for $t > \bar{t}$. On the other hand, the fact that

$$\int_{\mathbb{R}} F(u) dt < +\infty$$

implies that

$$\liminf_{t\to-\infty} u(t) = -1 \text{ and } \limsup_{t\to+\infty} u(t) = 1.$$

In fact we have $u(\pm\infty) = \pm 1$ as a consequence of the following claim.

Claim. Let $\varepsilon \in]0,1[$ *and*

$$\beta_\varepsilon := \min\left\{F(z) \mid \ z \in [1-\varepsilon, 1-\frac{\varepsilon}{2}] \cup [-1+\frac{\varepsilon}{2}, -1+\varepsilon]\right\}. \tag{1.11}$$

If $u \in \Gamma$ *has the property that there exist* $t_1, t_2 \in \mathbb{R}$ *such that* $u(t_1) = 1-\frac{\varepsilon}{2}$ *and* $u(t_2) = 1-\varepsilon$ *(or* $u(t_1) = -1+\frac{\varepsilon}{2}$ *and* $u(t_2) = -1+\varepsilon$*), then we have*

$$\mathcal{I}(u) \geq \int_{t_1}^{t_2} \left(\frac{u'^2}{2} + F(u)\right) dt \geq \frac{\varepsilon\sqrt{\beta_\varepsilon}}{\sqrt{2}}.$$

We may assume $t_1 < t_2$ and $1-\varepsilon \leq u(t) \leq 1-\frac{\varepsilon}{2}$ for $t \in [t_1, t_2]$ (or $-1+\frac{\varepsilon}{2} \leq u(t) \leq -1+\varepsilon$). By means of Schwarz's inequality, we infer that

$$\left|\frac{\varepsilon}{2}\right| = |u(t_2) - u(t_1)| \leq \sqrt{t_2 - t_1}\, \|u'\|_{L^2(t_1,t_2)}.$$

Using the positivity of the integrand, we now deduce that

$$\mathcal{I}(u) \geq \int_{t_1}^{t_2} \left(\frac{u'^2}{2} + F(u)\right) dt \geq \frac{\varepsilon^2}{8(t_2 - t_1)} + \beta_\varepsilon(t_2 - t_1)$$

and the claim follows from the elementary inequality $\dfrac{a^2}{x} + b^2 x \geq 2ab$, which holds for all $x \geq 0$.

Assume now $u(\pm\infty) \neq \pm 1$. Then there exist $\delta > 0$ and infinitely many disjoint intervals $[t_1, t_2]$ in the conditions of the claim with $\varepsilon = \delta$, implying $\mathcal{I}(u) = +\infty$ and contradicting (1.10).

Conclusion - It follows from the previous steps that $u \in \Gamma$ and

$$\mathcal{I}(u) = \min_\Gamma \mathcal{I}.$$

The fact that u is a heteroclinic solution of (1.5) follows from Proposition 1.1. □

1.2.3. The Periodic Case. We next apply the above method to a less trivial situation: let us consider the second order non-autonomous differential equation

$$u'' - a(t) f(u) = 0, \tag{1.12}$$

where $f \in C(\mathbb{R})$ satisfies assumptions (A1), (A2) and $a \in L^\infty(\mathbb{R})$ is bounded away from zero, i.e. $a(t) \geq a_1$ for some $a_1 > 0$ and all $t \in \mathbb{R}$. We still look for a heteroclinic connection between the equilibria -1 and $+1$. In the absence of a conservation law, the variational argument appears as a natural device. So we now consider the functional

$$\mathcal{J}(u) := \int_{\mathbb{R}} \left(\frac{u'^2}{2} + a(t)F(u)\right) dt \tag{1.13}$$

and seek conditions that allow to minimize it in Γ. We still assume that F is extended by 0 outside the interval $]-1,1[$. If $a(t)$ is a T-periodic function, then the proof of Theorem 1.2 can be easily adapted.

Theorem 1.3. *Assume that $a \in L^\infty(\mathbb{R})$ is a T-periodic function such that $a(t) \geq a_1$ for some $a_1 > 0$ and all $t \in \mathbb{R}$, $f \in C([-1,1])$ satisfies assumptions* (A1) *and* (A2). *Then the functional $\mathcal{J} : \Gamma \to \mathbb{R} \cup \{+\infty\}$ defined by* (1.13) *and* (1.9), *achieves a minimum. A minimizer $u \in \Gamma$ is a heteroclinic solution of* (1.12) *connecting -1 to $+1$ and satisfying $-1 \leq u(t) \leq 1$ for all $t \in \mathbb{R}$.*

Observe first that the Claim of the third step in the proof of Theorem 1.2 has a straightforward extension. As it is useful in a further situation, we state it as a lemma. From now on we also use extensively the following notation. If $\mathcal{I}$ is a functional defined from a Lagrangian $L(t,u,u')$ and $S \subset \mathbb{R}$, we set up

$$\mathcal{I}|_S(u) := \int_S L(t,u,u')\,dt.$$

If S is an interval, say $[a,b]$, then we also use the notation

$$\mathcal{I}|_a^b(u) := \mathcal{I}|_S(u).$$

Lemma 1.4. *Assume $a \in L^\infty(\mathbb{R})$ is such that $a(t) \geq a_1$ for some $a_1 > 0$ and all $t \in \mathbb{R}$ and $f \in C([-1,1])$ satisfies assumptions* (A1) *and* (A2). *Let $\varepsilon \in]0,1[$ and β_ε be defined by* (1.11). *If $u \in \Gamma$ has the property that there exist $t_1, t_2 \in \mathbb{R}$ such that $u(t_1) = 1 - \frac{\varepsilon}{2}$ and $u(t_2) = 1 - \varepsilon$ (or $u(t_1) = -1 + \frac{\varepsilon}{2}$ and $u(t_2) = -1 + \varepsilon$), then we have*

$$\mathcal{J}(u) \geq \mathcal{J}|_{t_1}^{t_2}(u) \geq \frac{\varepsilon\sqrt{a_1\beta_\varepsilon}}{\sqrt{2}}.$$

While we cannot completely mimic the powerful modification arguments used in the autonomous case, the periodicity of a allows to localize a transition from a small neighborhood of -1 to a small one of $+1$. Indeed, keeping the notations of the proof of Theorem 1.2, the left-hand side of the sequence of intervals $[s_n, t_n]$ may be assumed to belong to $[0,T[$ by means of a time translation of a length multiple of T. The remaining arguments are then similar except in the way to ensure that the quasi-minimizers stay close to ± 1 once they enter sufficiently small neighborhoods of these points. The modifications can be made according to the following lemma.

Lemma 1.5. *Assume $f \in C([-1,1])$ satisfies assumptions* (A1) *and* (A2) *and $a \in L^\infty(\mathbb{R})$ is a positive function. Let $\varepsilon \in]0,1[$ and define $\Gamma_{\bar{t},1-\varepsilon}$ as the set of functions $u \in C\left([\bar{t},+\infty[\right)$ such that $u(\bar{t}) = 1-\varepsilon$, $u(+\infty) = 1$ and $u' \in L^2([\bar{t},+\infty[)$. Then, there exists $v \in \Gamma_{\bar{t},1-\varepsilon}$ such that $1-\varepsilon \leq v(t) \leq 1$ for all $t \geq \bar{t}$ and*

$$\mathcal{J}|_{\bar{t}}^{+\infty}(v) \leq \frac{\varepsilon^2}{2} + a_2 \max_{z\in[1-\varepsilon,1]} F(z),$$

where $a_2 > 0$ is an upper bound for $a(t)$. Analogously, if we denote by $\Gamma_{\bar{t},-1+\varepsilon}$ the set of functions $u \in C(]-\infty,\bar{t}])$ satisfying $u(\bar{t}) = -1+\varepsilon$,

$u(-\infty) = -1$ and $u' \in L^2(]-\infty, \bar{t}\,])$, then there exists $w \in \Gamma_{\bar{t},-1+\varepsilon}$ such that $-1 \le w(t) \le -1+\varepsilon$ for all $t \le \bar{t}$ and

$$\mathcal{J}|_{-\infty}^{\bar{t}}(w) \le \frac{\varepsilon^2}{2} + a_2 \max_{z\in[-1,-1+\varepsilon]} F(z).$$

PROOF. It suffices to compute $\mathcal{J}|_{\bar{t}}^{+\infty}(v)$, where

$$v(t) = \begin{cases} 1-\varepsilon+\varepsilon(t-\bar{t}), & \text{if } \bar{t} \le t \le \bar{t}+1 \\ 1, & \text{if } t \ge \bar{t}+1. \end{cases}$$

$\square$

PROOF OF THEOREM 1.3. Let us fix $\bar{\varepsilon} \in]0,1[$ and choose $\varepsilon > 0$ sufficiently small to satisfy $\varepsilon < \bar{\varepsilon}/2$ and

$$\frac{\varepsilon^2}{2} + a_2\mu_\varepsilon < \frac{\bar{\varepsilon}\sqrt{a_1\beta_{\bar{\varepsilon}}}}{\sqrt{2}},$$

where $a_2 > 0$ is an upper bound of $a(t)$,

$$\mu_\varepsilon := \max_{z\in[-1,-1+\varepsilon]\cup[1-\varepsilon,1]} F(z) \tag{1.14}$$

and $\beta_{\bar{\varepsilon}}$ is defined according to (1.11). Let $(u_n)_n \subset \Gamma$ be a minimizing sequence for $\mathcal{J}$.

Step 1 - Modification of the minimizing sequence. Arguing as in the proof of Theorem 1.2, we may assume without loss of generality that $-1 \le u_n(t) \le 1$ for all $t \in \mathbb{R}$. Define the interval $[s_n, t_n]$ with respect to u_n and ε as in the proof of Theorem 1.2. It follows from Lemma 1.4 that if $u_n(t) \notin]1-\bar{\varepsilon}, 1]$ for some $t > t_n$, then

$$\mathcal{J}|_{t_n}^{+\infty}(u_n) \ge \frac{\bar{\varepsilon}\sqrt{a_1\beta_{\bar{\varepsilon}}}}{\sqrt{2}}.$$

On the other hand, Lemma 1.5 provides a function $v_n \in \Gamma_{t_n,1-\varepsilon}$ such that

$$\mathcal{J}|_{t_n}^{+\infty}(v_n) \le \frac{\varepsilon^2}{2} + a_2 \max_{z\in[1-\varepsilon,1]} F(z) \le \frac{\varepsilon^2}{2} + a_2\mu_\varepsilon.$$

Hence, according to the choice of ε, and modifying $u_n(t)$ in $[t_n, +\infty[$ if necessary, we may assume that

$$1-\bar{\varepsilon} \le u_n(t) \le 1 \text{ if } t \ge t_n$$

and

$$\mathcal{J}|_{t_n}^{+\infty}(u_n) \le \frac{\varepsilon^2}{2} + a_2\mu_\varepsilon.$$

For similar reasons, we may assume that

$$-1 \le u_n(t) \le -1+\bar{\varepsilon} \text{ if } t \le s_n,$$

and

$$\mathcal{J}|_{-\infty}^{s_n}(u_n) \leq \frac{\varepsilon^2}{2} + a_2\mu_\varepsilon.$$

Step 2 - Convergence. As in the proof of Theorem 1.2, $t_n - s_n$ is bounded. Due to the T- periodicity of a and translating u_n if necessary, we may suppose that $s_n \in [0, T[$. It now follows that along some subsequence

$$u_n \overset{C_{\mathrm{loc}}(\mathbb{R})}{\longrightarrow} u,\ u_n' \overset{L^2(\mathbb{R})}{\rightharpoonup} u',\ s_n \to \bar{s} \in [0, T],\ t_n \to \bar{t}$$

and in particular

$$-1 \leq u(t) \leq -1 + \varepsilon \ \text{ for all } t \leq \bar{s}, \quad 1 - \varepsilon \leq u(t) \leq 1 \ \text{ for all } t \geq \bar{t}.$$

Conclusion - Arguing as in the proof of Theorem 1.2, it is easily shown that $u \in \Gamma$ and

$$\mathcal{J}(u) = \inf_{\Gamma} \mathcal{J}.$$

The fact that u is a solution of (1.12) follows from the arguments of Proposition 1.1. To show that $u'(\pm\infty) = 0$, we argue by contradiction. Assume that there exist $\tau_n \to +\infty$ and $\Delta > 0$ with $|u'(\tau_n)| \geq \Delta$. As from (1.12) we have $u''(\pm\infty) = 0$, we may assume that, for n sufficiently large,

$$|u'(t)| \geq \frac{\Delta}{2} \ \text{ for all } t \in [\tau_n, \tau_n + 1].$$

It the follows that

$$|u(\tau_n) - u(\tau_n + 1)| \geq \frac{\Delta}{2},$$

which is impossible since $u(\pm\infty) = \pm 1$. □

1.2.4. The Bounded Case. If a does not possess any symmetry or periodicity property then we have to face a possible loss of compactness as the interval $[s_n, t_n]$ could escape to $+\infty$ or $-\infty$ so that the weak limit of u_n could either be one equilibrium or a homoclinic solution. One way to avoid such behaviours is to impose some coercivity assumption on the function a at $\pm\infty$. In some sense this penalize transitions appearing far away from 0. The following theorem is related to the results proved by C.-N. Chen and S.-Y. Tzeng [**32**].

THEOREM 1.6. *Assume that $f \in C([-1, 1])$ satisfies* (A1) *and* (A2). *Let $a \in L^\infty(\mathbb{R}, \mathbb{R})$ be such that $a_1 \leq a(t) \leq a_2$ for some a_1, $a_2 > 0$ and all $t \in \mathbb{R}$. If in addition*

$$\lim_{|t| \to +\infty} a(t) = a_2$$

and $a(t) < a_2$ in some set of positive measure, then (1.12) *has a heteroclinic solution from -1 to 1.*

Proof. We introduce the family of functionals

$$\mathcal{I}_\alpha(u) := \int_{\mathbb{R}} \left(\frac{u'^2}{2} + \alpha F(u) \right) dt,$$

where $\alpha > 0$ and the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\varphi(\alpha) := \min_{u\in\Gamma} \mathcal{I}_\alpha(u). \tag{1.15}$$

According to Theorem 1.2, $\mathcal{I}_\alpha$ has a minimum $u_\alpha \in \Gamma$ for each $\alpha > 0$ so that we have $\varphi(\alpha) = \mathcal{I}_\alpha(u_\alpha)$.

Claim 1. The function φ defined by (1.15) *is a strictly increasing, continuous function in $\mathbb{R}^+$.* The first statement is obvious. The second is a consequence of the inequalities

$$\varphi(\alpha) + (\beta - \alpha) \int_{\mathbb{R}} F(u_\beta)\, dt \le \varphi(\beta) \le \varphi(\alpha) + (\beta - \alpha) \int_{\mathbb{R}} F(u_\alpha)\, dt$$

together with the fact that, when β is close to α, the term

$$\int_{\mathbb{R}} F(u_\beta)\, dt$$

is uniformly bounded from above.

It clearly follows from the assumptions that

$$\inf_\Gamma \mathcal{J} < \varphi(a_2)$$

and we therefore infer from Claim 1 that there exists $\alpha \in]0, a_2[$ such that $\inf_\Gamma \mathcal{J} < \varphi(\alpha)$. We then fix $\bar\varepsilon \in]0,1[$ and choose $\varepsilon > 0$ sufficiently small to satisfy $\varepsilon < \bar\varepsilon/2$ and

$$(1 + \frac{a_2}{a_1})(\frac{\varepsilon^2}{2} + a_2\mu_\varepsilon) < \min \left(\frac{\bar\varepsilon\sqrt{a_1\beta_{\bar\varepsilon}}}{\sqrt{2}}, \frac{\varphi(\alpha) - \inf_\Gamma \mathcal{J}}{2} \right), \tag{1.16}$$

where $\beta_{\bar\varepsilon}$ is defined according to (1.11) and μ_ε by (1.14).

Let $(u_n)_n \subset \Gamma$ be a minimizing sequence for $\mathcal{J}$ and define the interval $[s_n, t_n]$ with respect to u_n and ε as in the proof of Theorem 1.2. Arguing as in the proof of Theorem 1.3, we may assume that

$$-1 \le u_n(t) \le -1 + \bar\varepsilon \text{ if } t \le s_n,\ \ 1 - \bar\varepsilon \le u_n(t) \le 1 \text{ if } t \ge t_n$$

and

$$\sup \left(\mathcal{J}|_{-\infty}^{s_n}(u_n),\ \mathcal{J}|_{t_n}^{+\infty}(u_n) \right) \le \frac{\varepsilon^2}{2} + a_2\mu_\varepsilon.$$

Claim 2. The interval $t_n - s_n$ is uniformly bounded. To prove this claim, we show that s_n is bounded from above and t_n is bounded from below. Let us prove this for s_n, the proof for t_n being similar. If the claim is not true there exists $s_0 \in \mathbb{R}$ such that $a(t) \geq \alpha$ whenever $t \geq s_0$ so that $a(t) \geq \alpha$ whenever $t \geq s_n$ for large n. We then compute for such n,

$$\begin{aligned}\mathcal{J}(u_n) &\geq \int_{s_n}^{+\infty} \left(\frac{u_n'^2}{2} + \alpha F(u_n) \right) dt \\ &\geq \varphi(\alpha) - \frac{a_2}{a_1} \left(\frac{\varepsilon^2}{2} + a_2 \mu_\varepsilon \right)\end{aligned}$$

and by our choice of ε in (1.16), we infer that

$$\mathcal{J}(u_n) \geq \frac{\varphi(\alpha) + \inf_\Gamma \mathcal{J}}{2},$$

a contradiction with the assumption $\mathcal{J}(u_n) \to \inf_\Gamma \mathcal{J}$.

The remaining of the proof is now similar to that of Theorem 1.3. □

1.3. Reversible Hamiltonian Systems

In this section we consider a basic situation for systems, adapted from the results of P. H. Rabinowitz [**87**]. We focus on the autonomous system

$$u'' - \nabla V(u) = 0, \tag{1.17}$$

where $u = (u_1, \ldots, u_N)$ and $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is a non-negative potential with several isolated equilibria at the minimum level, which we assume to be zero. We denote by $\mathcal{M}$ the set of global minimizers of V, i.e. $\mathcal{M} := V^{-1}(0)$. Precisely, we assume

(A3) $V(u) \geq 0$ for all $u \in \mathbb{R}$, $\mathcal{M}$ contains at least two points and

$$\inf \{ |\xi_1 - \xi_2| \mid \xi_1, \xi_2 \in \mathcal{M} \text{ and } \xi_1 \neq \xi_2 \} > 0;$$

(A4) for each $\varepsilon > 0$, $\inf \{ V(u) \mid \operatorname{dist}(u, \mathcal{M}) \geq \varepsilon \} > 0$.

As in the previous section, we investigate the existence of heteroclinic connections between elements of $\mathcal{M}$. A solution u of (1.17) such that

$$u(-\infty) = \xi, \quad u(+\infty) = \eta$$

for some $\xi, \eta \in \mathcal{M}$, $\xi \neq \eta$ and

$$u'(\pm\infty) = 0,$$

is called a heteroclinic solution of (1.17) from ξ to η or a heteroclinic connection between ξ and η. We say also that u starts at ξ and ends at η.

Clearly, if $u(t)$ is a heteroclinic of (1.17) from ξ to η, then for every $c \in \mathbb{R}$ $u(t+c)$ is also a heteroclinic from ξ to η, and $u(-t)$ is a heteroclinic from η to ξ.

Let us fix an element in $\mathcal{M}$, which we may assume without loss of generality to be 0. Then, we search for heteroclinics from 0 to some $\xi \in \mathcal{M}\backslash\{0\}$. They appear as suitable minimizers of the functional

$$\mathcal{K}(u) := \int_{\mathbb{R}} \left(\frac{|u'|^2}{2} + V(u) \right) dt, \tag{1.18}$$

where $|u'|^2$ denote the square of the norm of the vector u', i.e.

$$|u'|^2 = {u'_1}^2 + \ldots + {u'_N}^2.$$

Given $\xi \in \mathcal{M}\backslash\{0\}$, it is natural to consider the class of vector functions

$$\Gamma(\xi) := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N) \mid \ u(-\infty) = 0,\ u(+\infty) = \xi \right\}.$$

Let us set

$$c(\xi) := \inf_{u \in \Gamma(\xi)} \mathcal{K}(u).$$

THEOREM 1.7. *Let $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfy assumptions* (A3) *and* (A4), *with $0 \in \mathcal{M}$. Then, the system* (1.17) *has a heteroclinic solution starting at* 0, *and another one ending at* 0.

The proof requires some preliminaries. The next lemma is proved with a computation quite similar to the proof of Lemma 1.4. For any interval $[a, b] \subset \mathbb{R}$, we still use the notation

$$\mathcal{K}|_a^b(u) = \int_a^b \left(\frac{|u'|^2}{2} + V(u) \right) dt.$$

LEMMA 1.8. *Assume that $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies assumptions* (A3) *and* (A4). *Let $\varepsilon > 0$, and $u \in H^1((t_1, t_2), \mathbb{R}^N)$ be such that*

$$\mathrm{dist}(u(t), \mathcal{M}) \geq \varepsilon \quad \textit{for all } t \in [t_1, t_2].$$

Then

$$\mathcal{K}|_{t_1}^{t_2}(u) \geq \sqrt{2\alpha_\varepsilon}|u(t_2) - u(t_1)|,$$

where $\alpha_\varepsilon := \inf\{V(u) \mid \mathrm{dist}(u, \mathcal{M}) \geq \varepsilon\} > 0$.

We then prove that any function of $H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ with a finite action, do have limits at $\pm\infty$. Moreover, these are elements of $\mathcal{M}$.

LEMMA 1.9. *Assume that $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies* (A3) *and* (A4). *Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ be such that $\mathcal{K}(u) < +\infty$. Then $u(-\infty)$, $u(+\infty)$ exist and belong to $\mathcal{M}$.*

PROOF. The proof follows from the two following claims.

Claim 1. There exist sequences $t_n \to -\infty$, $s_n \to +\infty$ such that $u(t_n)$ and $u(s_n)$ belong to $\mathcal{M}$. As the term

$$\int_{\mathbb{R}} V(u(t))\, dt$$

is bounded, we have

$$\liminf_{t\to-\infty} V(u(t)) = \liminf_{t\to+\infty} V(u(t)) = 0.$$

Hence, by assumption (A4), there exist sequences $t_n \to -\infty$, $s_n \to +\infty$ such that

$$\lim_{n\to+\infty} \operatorname{dist}(u(t_n), \mathcal{M}) = \lim_{n\to+\infty} \operatorname{dist}(u(s_n), \mathcal{M}) = 0.$$

Claim 2. The limits exist and belong to $\mathcal{M}$. Let us prove that $u(+\infty)$ exists and belongs to $\mathcal{M}$, a similar argument applies for the limit at $-\infty$. If the limit does not exist we are able to find

$$0 < \varepsilon < \gamma := \frac{1}{3} \inf\{|\xi_1 - \xi_2| \mid \xi_1 \neq \xi_2 \text{ and } \xi_1, \xi_2 \in \mathcal{M}\} \tag{1.19}$$

and disjoint intervals $[\tau_n, \sigma_n]$ such that

$$\operatorname{dist}(u(\tau_n), \mathcal{M}) = \varepsilon, \quad \operatorname{dist}(u(\sigma_n), \mathcal{M}) = 2\varepsilon$$

and

$$\operatorname{dist}(u(t), \mathcal{M}) \geq \varepsilon$$

for all $t \in [\tau_n, \sigma_n]$. But then, through the use of Lemma 1.8, we infer that

$$\begin{aligned}\mathcal{K}(u) &\geq \sum_{n=0}^{\infty} \mathcal{K}|_{\tau_n}^{\sigma_n}(u) \\ &\geq \sum_{n=0}^{+\infty} \sqrt{2\alpha_\varepsilon} \min(\varepsilon, \gamma - \varepsilon) = +\infty,\end{aligned}$$

which is a contradiction. □

LEMMA 1.10. *Assume $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies assumptions* (A3) *and* (A4)*. If the set $\mathcal{M}$ is unbounded, then*

$$\lim_{|\xi|\to+\infty} c(\xi) = +\infty.$$

PROOF. It suffices to prove that, given $M \geq 0$, there exists another number $N = N(M)$ such that $\mathcal{K}(u) \leq M$ and $u \in \Gamma(\xi)$ implies $|\xi| \leq N$.

Fix $\varepsilon > 0$ as in (1.19) and define α_ε as in Lemma 1.8. For u as stated, if $|u(a) - \xi_a| = \varepsilon = |u(b) - \xi_b|$, $\xi_a \neq \xi_b$, and for all $t \in [a, b]$, $\operatorname{dist}(u(t), \mathcal{M}) \geq \varepsilon$, we say that $[a, b]$ is a transition interval from $\xi_a \in \mathcal{M}$ to $\xi_b \in \mathcal{M}$.

Claim 1. The length of a transition interval is a priori bounded. Indeed, this follows easily from assumption (A4). If $[a, b]$ is a transition interval, then we have

$$M \geq \int_a^b V(u(t))\, dt \geq \alpha_\varepsilon (b - a).$$

Claim 2. u visits a finite number of minima. It follows from Claim 1 that u cannot stay too much time bounded away from $\mathcal{M}$. Therefore, we are able to find disjoint transition intervals $[t_i, s_i]$, from ξ_i to η_i, $i = 1, \ldots, k$ such that $\xi_1 = 0$, $\xi_{j+1} = \eta_j$ and $\eta_k = \xi$. We may assume without loss of generality that the ξ_j's are all distinct. Then, Lemma 1.8 implies that

$$\mathcal{K}(u) \geq \sum_{i=1}^{k} \sqrt{2\alpha_\varepsilon}\gamma = k\sqrt{2\alpha_\varepsilon}\gamma.$$

Hence, the number k of such intervals is a priori bounded

$$k \leq k_0(M) := \frac{M}{\gamma\sqrt{2\alpha_\varepsilon}},$$

so that Claim 2 holds.

Using Schwarz's inequality and the estimate of Claim 1, we now deduce that

$$|u(t_i) - u(s_i)| \leq \|u'\|_{L^2(t_i,s_i)}\sqrt{s_i - t_i} \leq M\sqrt{\frac{2}{\alpha_\varepsilon}}.$$

We therefore conclude that

$$\begin{aligned}
|\xi| &= |u(+\infty) - u(-\infty)| \\
&= |\xi - u(s_k) + u(s_k) - u(t_k) + \ldots + u(s_2) - u(t_1) + u(t_1) - \xi_1| \\
&\leq 2\varepsilon + \sum_{i=2}^{k} |u(s_i) - u(t_i)| + \sum_{i=2}^{k-1} |u(t_{i+1}) - u(s_i)| \\
&\leq 2\varepsilon + (k_0(M) - 1)M\sqrt{\frac{2}{\alpha_\varepsilon}} + 2\varepsilon(k_0(M) - 2),
\end{aligned}$$

so that we derive an upper bound for $|\xi|$, which depends only on M. $\square$

We now turn to the proof of the existence of a heteroclinic solution starting at 0. This solution achieves the minimum level of the action among all possible connections between 0 and another element of $\mathcal{M}$.

Proof of Theorem 1.7. By reversing the time, it is sufficient to prove that the system (1.17) possesses a heteroclinic starting at 0. To show this, we prove the existence of a heteroclinic solution at the minimum level of $\mathcal{K}$ among all possible connections starting from 0. It follows from assumption (A3) and Lemma 1.10, if $\mathcal{M}$ is not finite, that

$$c := \min_{\zeta \in \mathcal{M}\setminus\{0\}} c(\zeta)$$

exists, since a bounded domain intersects $\mathcal{M}$ in a finite set. Hence, there exists $\xi \in \mathcal{M}\setminus\{0\}$ such that $c = c(\xi)$. Now let $(u_n)_n \subset \Gamma(\xi)$ be a minimizing sequence of $\mathcal{K}$, i.e. for all $n \in \mathbb{N}$, $u_n \in \Gamma(\xi)$ and

$$\mathcal{K}(u_n) \to c(\xi) = c.$$

Step 1 - Convergence. Let $\gamma > 0$ be defined by (1.19) and fix $\varepsilon > 0$ in such a way that $\varepsilon < \gamma$. By translation invariance, we may assume that

$$|u_n(0)| = \varepsilon, \quad |u_n(t)| \le \varepsilon \quad \text{for all } t \le 0 \text{ and all } n \in \mathbb{N}. \tag{1.20}$$

From the boundedness of $\mathcal{K}(u_n)$, it follows immediately that $(u_n')_n$ is bounded for the L^2-norm, and the arguments of the proof of Lemma 1.10 imply that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(\mathbb{R})} < +\infty.$$

Hence, for each finite interval $[a, b]$, we deduce that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{H^1(a,b)} < +\infty.$$

By means of the diagonal procedure, we may now extract a subsequence, still labelled $(u_n)_n$, and find a function $u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ such that

$$u_n \overset{C_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)}{\longrightarrow} u \quad \text{and} \quad u_n' \overset{L^2(\mathbb{R},\mathbb{R}^N)}{\rightharpoonup} u'.$$

The weak lower semi-continuity of the norm and Fatou's lemma imply

$$\mathcal{K}(u) \le \liminf_{n\to+\infty} \mathcal{K}(u_n) = c.$$

Step 2 - $u \in \Gamma(\xi)$. By Lemma 1.9, $u(\pm\infty)$ exist and belong to $\mathcal{M}$. From the uniform estimate in (1.20), we infer, using pointwise convergence, that $u(-\infty) = 0$. We now prove that $u(+\infty) = \xi$. This follows from the following two claims.

Claim 1. $u(+\infty) \ne 0$. Suppose by contradiction that $u(+\infty) = 0$. Fix $\delta > 0$ satisfying $4\delta < \varepsilon$ and

$$2\delta^2 + \sup_{B_{2\delta}(0)} V < \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}}, \tag{1.21}$$

where $\alpha_{\varepsilon/2}$ is defined as in Lemma 1.8. Let $t_\delta > 0$ be such that for all $t \ge t_\delta$, $u(t) \in B_\delta(0)$. Then, we also have $u_m(t_\delta) \in B_{2\delta}(0)$ for all $m \in \mathbb{N}$ sufficiently large. As $|u_m(0)| = \varepsilon$, Lemma 1.8 implies that

$$\mathcal{K}(u_m) \ge \frac{\varepsilon}{2}\sqrt{2\alpha_{\varepsilon/2}} + \mathcal{K}|_{t_\delta}^{+\infty}(u_m). \tag{1.22}$$

Define a new function $U_m \in \Gamma(\xi)$ by

$$U_m(t) := \begin{cases} 0, & \text{if } t \le t_\delta - 1, \\ (t - t_\delta + 1)u_m(t_\delta), & \text{if } t_\delta - 1 \le t \le t_\delta, \\ u_m(t), & \text{if } t \ge t_\delta. \end{cases}$$

We clearly have the estimate

$$\mathcal{K}(U_m) \le \frac{1}{2}(2\delta)^2 + \sup_{B_{2\delta}(0)} V + \mathcal{K}|_{t_\delta}^{+\infty}(u_m). \tag{1.23}$$

From (1.21), (1.22) and (1.23) we conclude that

$$\mathcal{K}(U_m) \le \mathcal{K}(u_m) - \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}}\,,$$

which yields a contradiction since

$$c \le \limsup_{m\to+\infty} \mathcal{K}(U_m) \le c - \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}} < c.$$

Claim 2. $u(+\infty) = \xi$. From what precedes, we already know that $u(+\infty) = \eta$ for some $\eta \in \mathcal{M} \setminus \{0\}$. Suppose, by contradiction, that $u(+\infty) = \eta \in \mathcal{M}\setminus\{0, \xi\}$. We then fix $\varepsilon > 0$ and $\delta > 0$ in such a way that

$$\varepsilon < \frac{\gamma}{2}, \quad \delta < \frac{\varepsilon}{2} \quad \text{and} \quad 2\delta^2 + \sup_{B_{2\delta}(\eta)} V < \frac{\gamma - 2\varepsilon}{2}\sqrt{2\alpha_\varepsilon}.$$

Let $t_\delta > 0$ be such that $u_m(t_\delta) \in B_{2\delta}(\eta)$ for all m sufficiently large. We introduce the new function $U_m \in \Gamma(\eta)$ defined by

$$U_m(t) := \begin{cases} u_m(t), & \text{if } t \le t_\delta, \\ (1 - t + t_\delta)u_m(t_\delta) + (t - t_\delta)\eta, & \text{if } t_\delta \le t \le t_\delta + 1, \\ \eta, & \text{if } t \ge t_\delta + 1. \end{cases}$$

As $u_m \in \Gamma(\xi)$, there exist $\underline{t}_m < \bar{t}_m$ with $t_\delta < \underline{t}_m$, such that

$$|u_m(\underline{t}_m) - \eta| = \varepsilon, \quad \operatorname{dist}(u_m(\bar{t}_m), \mathcal{M} \setminus \{\eta\}) = \varepsilon$$

and

$$\operatorname{dist}(u_m(t), \mathcal{M}) \ge \varepsilon \text{ for all } t \in [\underline{t}_m, \bar{t}_m].$$

Hence, we deduce from Lemma 1.8 that

$$\mathcal{K}(u_m) \ge \mathcal{K}|_{-\infty}^{t_\delta}(u_m) + (\gamma - 2\varepsilon)\sqrt{2\alpha_\varepsilon}.$$

On the other hand, we infer that

$$\mathcal{K}(U_m) \le 2\delta^2 + \sup_{B_{2\delta}(\eta)} V + \mathcal{K}|_{-\infty}^{t_\delta}(u_m)$$

and therefore conclude that

$$\mathcal{K}(U_m) \le \mathcal{K}(u_m) - \frac{\gamma - 2\varepsilon}{2}\sqrt{2\alpha_\varepsilon}.$$

It now follows that

$$\limsup_{m\to+\infty} \mathcal{K}(U_m) \le c - \frac{\gamma - 2\varepsilon}{2}\sqrt{2\alpha_\varepsilon} < c,$$

which again contradicts the definition of the level c.

Conclusion - Having shown that $u \in \Gamma(\xi)$, it follows that $c = \mathcal{K}(u)$ is a minimum of $\mathcal{K}$ in $\Gamma(\xi)$. The usual elementary argument of the Calculus of Variations shows that u is a solution of (1.17). At last we must check that $u'(\pm\infty) = 0$. For an autonomous system, this is particularly simple. Indeed, since u is a solution of (1.17), it satisfies the energy identity

$$\frac{|u'|^2}{2} + V(u) = K$$

for some constant $K \geq 0$, and it is easy to see that $K = 0$. □

In presence of symmetries, something else can be said. Let us consider the important case where V is periodic in each coordinate. Through the use of a simple rescaling, we may assume without loss of generality that the period is the same for all coordinates and that the minima of V are located at the translates of 0:

(A5) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is a potential periodic in each variable u_i with period 1, $\min_{\mathbb{R}^N} V = 0$ and $\mathcal{M} = V^{-1}(0) = \mathbb{Z}^N$.

Clearly, condition (A5) implies that (A3) and (A4) hold, so that Theorem 1.7 applies to this class of potentials.

Theorem 1.11. *If* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *satisfies assumption* (A5)*, then for each* $\beta \in \mathcal{M}$ *there are at least* $2N$ *heteroclinic solutions of* (1.17) *starting from* β *and at least* $2N$ *heteroclinic solutions of* (1.17) *ending at* β*.*

Idea of the proof. We may assume that $\beta = 0$ and we keep the notation introduced before. Whenever the level $c(\xi)$ is attained for some $\xi \in \mathcal{M}\backslash\{0\}$, the minimizer is a heteroclinic from 0 to ξ. Let G be the subgroup of $\mathbb{Z}^N$ spanned by the elements $\xi \in \mathcal{M}\backslash\{0\}$ so that $c(\xi)$ is attained. It follows from Theorem 1.7 that $G \neq 0$. If $G \neq \mathbb{Z}^N$, we may select $\zeta \in \mathbb{Z}^N\backslash G$ satisfying

$$c(\zeta) = \min_{\zeta' \in \mathbb{Z}^N \backslash G} c(\zeta').$$

Then, mimicking the proof of Theorem 1.7, we are able to prove that $c(\zeta)$ is attained. The main difference is in the proof of Claim 2. When we take $\eta \in \mathcal{M}\backslash\{0, \zeta\}$ we have to allow the possibility that $\eta \in G$. But this implies that $\zeta - \eta \in \mathbb{Z}^N\backslash G$. The minimizing sequence (u_m) is then modified to a sequence in $\Gamma(\zeta - \eta)$, given by

$$U_m(t) := \begin{cases} 0, & \text{if } t \leq t_\delta - 1, \\ (t - t_\delta + 1)(u_m(t_\delta) - \eta), & \text{if } t_\delta - 1 \leq t \leq t_\delta, \\ u_m(t) - \eta, & \text{if } t \geq t_\delta. \end{cases}$$

The appropriate choice of δ and the fact that $V(u_m(t)-\eta)=V(u_m(t))$ lead to the usual contradiction with the choice of ζ. Therefore we conclude that $G=\mathbb{Z}^N$. This yields N $\mathbb{Z}$-independent elements ξ and N corresponding heteroclinics u form 0 to ξ. For each such ξ, $u(-t)-\xi$ is a heteroclinic from 0 to $-\xi$. $\square$

1.4. Periodic Hamiltonian Systems: Heteroclinic Chains

In this section we are interested in non-autonomous systems of the form

$$u'' - \nabla_u V(t,u) = 0, \tag{1.24}$$

where the potential is 1-periodic in the time variable as well as in each space variable u_i $(i=1,\dots,N)$. The results of the previous section carry over to this kind of systems provided the assumptions on the potential are adequately rephrased. Namely, $V(t,\cdot)$ must have equilibria at the minimum level of the potential that are independent of t.

We have seen that, under certain conditions, a given element of $\mathcal{M}$ (the set of equilibria at the minimum level of V) can be connected by a heteroclinic to some different elements of $\mathcal{M}$. We devote this section to the problem of connecting *two distinct given elements* $\xi,\eta\in\mathcal{M}$, by a *heteroclinic chain*, that is, a finite set of heteroclinics $\{v_0,\dots,v_j\}$ such that $v_0(-\infty)=\xi$, $v_{i+1}(-\infty)=v_i(+\infty)$ for $i=0,\dots,j-1$ and $v_j(+\infty)=\eta$. While we cannot guarantee, in general, that a heteroclinic from ξ to η exists, we show that under simple assumptions, the heteroclinic chain always exists.

Let us state the assumptions on V:

- **(A6)** $V\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ is 1-periodic in each variable t and u_i, $i=1,\dots,n$;
- **(A7)** $V(t,0)=0<V(t,u)$ for all $t\in\mathbb{R}$ and $u\in\mathbb{R}^N\setminus\mathbb{Z}^N$.

As already emphasized, we may clearly consider different given periods in each variable of V through the use of a simple rescaling. Note also that (A6) and (A7) imply that $\mathcal{M}=\{u\in\mathbb{R}^N:\ V(t,u)=0\}=\mathbb{Z}^N$ independently of t.

We easily recognize that Lemma 1.8 and Lemma 1.9 extend to the class of potentials we are considering by now. We only have to replace $V(u)$ by $V(t,u)$ and define $\alpha_\varepsilon := \inf\{V(t,u)\mid t\in\mathbb{R},\ \operatorname{dist}(u,\mathcal{M})\ge\varepsilon\}$.

As before, we base our approach on the variational structure of the equation. We define the functional

$$\mathcal{Q}(u) := \int_{\mathbb{R}} \left(\frac{|u'|^2}{2} + V(t,u)\right) dt. \tag{1.25}$$

For each given pair of elements $\xi,\eta\in\mathcal{M}$, consider the class of functions

$$\Gamma(\xi,\eta)=\{u\in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)\mid\ u(-\infty)=\xi,\ u(+\infty)=\eta\}.$$

We then set

$$c(\xi,\eta) := \inf_{u\in\Gamma(\xi,\eta)} \mathcal{Q}(u). \tag{1.26}$$

The following statement is a particular case of a result of T. O. Maxwell [**68**].

THEOREM 1.12. *Assume $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ satisfies* (A6) *and* (A7). *For each pair $\xi,\eta \in \mathcal{M}$ $(\xi \neq \eta)$ there exists a finite set*

$$\{w_0, w_1, \ldots, w_j\} \subset \mathcal{M}$$

with $w_0 = \xi$, $w_j = \eta$, and a corresponding family of heteroclinic solutions $v^i \in \Gamma(w_i, w_{i+1})$ of (1.24), *with*

$$\mathcal{Q}(v^i) = c(w_i, w_{i+1}), \quad \textit{for } i = 0, \ldots, j-1$$

and

$$\sum_{i=0}^{j-1} \mathcal{Q}(v^i) = c(\xi,\eta).$$

In the proof of Theorem 1.12 we use repeatedly an argument similar to one introduced in the proof of Theorem 1.7, according to which some trajectories may be shortened in such a way that the new trajectory yields a lower value of the functional $\mathcal{Q}$. We state this in the following Lemma. Here, the constant γ introduced in (1.19) is $1/3$.

LEMMA 1.13. *Let $\theta \geq 1/3$ and $\eta > 0$ be such that $\eta < \theta/2$. Assume $u \in \Gamma(\xi,\eta)$, $w \in \mathcal{M}$ and $t_1 < t_2$ are such that*

$$|u(t_1) - w| \leq \eta, \; |u(t_2) - w| \leq \eta, \; \sup_{t\in[t_1,t_2]} |u(t) - w| \geq \theta. \tag{1.27}$$

Choose $0 < a < \min\{\frac{t_2-t_1}{2}, 1\}$ and define $U \in \Gamma(\xi,\eta)$ as

$$U(t) := \begin{cases} u(t), & \text{if } t \leq t_1 \text{ or } t \geq t_2, \\ \dfrac{t-t_1}{a} w + \left(1 - \dfrac{t-t_1}{a}\right) u(t_1), & \text{if } t_1 \leq t \leq t_1 + a, \\ w, & \text{if } t_1 + a \leq t \leq t_2 - a, \\ \left(\dfrac{t-t_2}{a} + 1\right) u(t_2) + \dfrac{t_2 - t}{a} w, & \text{if } t_2 - a \leq t \leq t_2. \end{cases}$$

Then

$$\mathcal{Q}(u) - \mathcal{Q}(U) \geq \frac{\theta}{2}\sqrt{2\alpha_{\theta/2}} - \left(\frac{\eta^2}{a} + 2 \max_{|z-w|\leq\eta} V(t,z)\right).$$

PROOF. The lemma is a straightforward consequence of the following estimates. On the one hand, we have

$$\mathcal{Q}|_{t_1}^{t_1+a}(U) \leq \frac{\eta^2}{2a} + \max_{|z-w|\leq\eta} V(t,z)$$

and the same inequality for $\mathcal{Q}|_{t_2-a}^{t_2}(U)$, while $\mathcal{Q}|_{t_1+a}^{t_2-a}(U) = 0$. On the other hand, a slight adaptation of the arguments of the proof of Lemma 1.8 yields the estimate

$$\mathcal{Q}|_{t_1}^{t_2}(u) \geq \frac{\theta}{2}\sqrt{2\alpha_{\theta/2}}.$$

$\square$

Proof of Theorem 1.12. Let $(u_n)_n \subset \Gamma(\xi, \eta)$ be a minimizing sequence for $\mathcal{Q}$.

Case 1. There exists $\varepsilon > 0$ and a subsequence, still denoted $(u_n)_n$, such that

$$\operatorname{dist}(u_n(\mathbb{R}),\ \mathcal{M}\backslash\{\xi, \eta\}) \geq \varepsilon.$$

By periodicity and a translation in the independent variable t we may assume that u_n leaves $B_\gamma(\xi)$ at $t = t_n^0 \in [0, 1[$, that is,

$$|u_n(t) - \xi| \leq \gamma \text{ for all } t \leq t_n^0,\ u_n(t_n^0) \in \partial B_\gamma(\xi). \tag{1.28}$$

Then, arguments quite similar to those of the proof of Theorem 1.2 allow us to assume that

$$u_n \overset{C_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)}{\longrightarrow} u \quad \text{and} \quad u_n' \overset{L^2(\mathbb{R},\mathbb{R}^N)}{\rightharpoonup} u'$$

for some function $u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$. We also deduce that $\mathcal{Q}(u) \leq c(\xi, \eta)$. Using (1.28) and the equivalent of Lemma 1.9, we infer that

$$\sup_{t\in\mathbb{R}} |u(t) - \xi| \geq \gamma \quad \text{and} \quad u(-\infty) = \xi.$$

We also observe that $u(+\infty) \in \{\xi, \eta\}$. If $u(+\infty) = \xi$, then for any $\delta \in]0, \gamma[$ sufficiently small, it is possible to find an interval $[t_1, t_2]$ having property (1.27) with $w = \xi$, $\eta = \delta/2$ and θ slightly larger than γ. Since $u_n \to u$ uniformly in $[t_1, t_2]$, (1.27) with $w = \xi$, $\eta = \delta$ and $\theta = \gamma$ is in fact true for u_n whenever n is sufficiently large. By means of Lemma 1.13, we are able to construct $U_n \in \Gamma(\xi, \eta)$ with some choice of $a > 0$ such that, for n large enough,

$$\mathcal{Q}(u_n) - \mathcal{Q}(U_n) \geq \frac{\gamma}{2}\sqrt{2\alpha_{\gamma/2}} - \left(\frac{\delta^2}{a} + 2 \max_{|z-w|\leq\delta} V(t, z)\right).$$

Clearly, we may suppose that δ has been chosen so small that the right-hand side of this inequality is strictly positive. We therefore infer that

$$\limsup_{n\to+\infty} \mathcal{Q}(U_n) < c(\xi, \eta),$$

contradicting the definition of $c(\xi, \eta)$. Hence, the level $c(\xi, \eta)$ is achieved at u and, using by now familiar arguments, we conclude that u is a solution of (1.24). This solution is heteroclinic form ξ to η, since it is easy to show, arguing as in Theorem 1.6, that $u'(\pm\infty) = 0$.

CASE 2. For all $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $k \geq m$

$$\operatorname{dist}(u_k(\mathbb{R}),\ \mathcal{M}\backslash\{\xi,\eta\}) < \varepsilon.$$

In this case, we cannot ensure the existence of a heteroclinic solution at the level $c(\xi,\eta)$. We prove instead the existence of a heteroclinic chain.

Step 1 - The links of the chain. Let us fix a sequence of numbers ε_k decreasing to 0. Then there exists a subset S of $\mathcal{M}\backslash\{\xi,\eta\}$ and a sequence of integers $(n_k)_k$ such that for $n \geq n_k$, $u_n(t)$ enters the neighborhood $B_{\varepsilon_k}(S)$. Arguing as in Lemma 1.10, it can be checked that S has to be finite. For each $w_i \in S \cup \{\xi,\eta\}$ and every $k \in \mathbb{N}$, we define the escape time ω_k^i meaning that u_{n_k} leaves the closure of $B_{\varepsilon_k}(w_i)$ at time $t = \omega_k^i$ and does not return to this ball in the future. Let us label the elements of S as $\{w_1,\ldots,w_{j-1}\}$ and set $w_0 = \xi$, $w_j = \eta$ in such a way that

$$\omega_k^0 < \omega_k^1 < \ldots < \omega_k^{j-1}.$$

Such a choice is possible at least for a subsequence we denote by $(u_k)_k$. Let us also consider the sequences $(\alpha_k^i)_k$ $(i = 1,\ldots,j)$ in such a way that at time $t = \alpha_k^i$, u_k reaches $\partial B_{\varepsilon_k}(w_i)$ for the first time.

Claim 1. For all k sufficiently large, we have $\omega_k^{i-1} < \alpha_k^i$, $i = 1,\ldots,j$. Otherwise, for a subsequence and some index $i_0 \in \{1,\ldots,j\}$, we have $\alpha_k^{i_0} < \omega_k^{i_0-1}$. Hence, we are able to find an interval $[t_{1_k}, t_{2_k}]$ satisfying the assumptions of Lemma 1.13 with $\delta = \varepsilon_k$, $\theta = \gamma$ and $w = w_{i_0}$. Modifying u_k and letting ε_k go to 0, this leads to a contradiction with the definition of $c(\xi,\eta)$.

Claim 2. There exists $\varepsilon_0 > 0$ such that for all $w \in \mathcal{M} \setminus \{w_{i-1}, w_i\}$, $i = 1,\ldots,j$ and k sufficiently large, if $\omega_k^{i-1} \leq t \leq \alpha_k^i$ then we have

$$|u_k(t) - w| \geq \varepsilon_0.$$

Otherwise, for some $i_0 \neq i-1$, i and a subsequence, $u_k([\omega_k^{i-1}, \alpha_k^i])$ meets $B_{\varepsilon_k}(w_{i_0})$, contradicting Claim 1.

Step 2 - The heteroclinic components of the chain. Let us define new elements $v_k^i \in \Gamma(w_i, w_{i+1})$ by

$$v_k^i(t) := \begin{cases} w_i, & t \leq \omega_k^i - 1, \\ (\omega_k^i - t)w_i + (t - \omega_k^i + 1)u_k(\omega_k^i), & \omega_k^i - 1 \leq t \leq \omega_k^i, \\ u_k(t), & \omega_k^i \leq t \leq \alpha_k^{i+1}, \\ (1 - t + \alpha_k^{i+1})u_k(\alpha_k^{i+1}) + (t - \alpha_k^{i+1})w_{i+1}, & \alpha_k^{i+1} \leq t \leq \alpha_k^{i+1}+1, \\ w_{i+1}, & t \geq \alpha_k^{i+1}+1. \end{cases}$$

It is clear that $\mathcal{Q}(v_k^i)$ is uniformly bounded. By periodicity and a translation in the independent variable t, we may assume without loss of generality that u_k is such that

$$|v^i_k(t) - w_i| \le \gamma \text{ for all } t \le t^i_k \in [0,1[\,,\ u_k(t^i_k) \in \partial B_\gamma(\xi).$$

Then, by now familiar arguments show that there exist $v^i \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ and a subsequence we still denote by $(v^i_k)_k$ with the properties

$$v^i_k \stackrel{C_{\text{loc}}(\mathbb{R},\mathbb{R}^N)}{\longrightarrow} v^i,\ (v^i_k)' \stackrel{L^2(\mathbb{R},\mathbb{R}^N)}{\rightharpoonup} (v^i)'.$$

Moreover, we infer that

$$|v^i(t) - w_i| \le \gamma \ \text{ for all } t \le t^i \in [0,1[$$

and

$$\sup_{t\in\mathbb{R}} |u(t) - w_i| \ge \gamma.$$

As a consequence of Claim 2 in Step 1 and the equivalent of Lemma 1.9, we also deduce that

$$v^i(-\infty) = w_i \ \text{ and } \ v^i(+\infty) \in \{w_i, w_{i+1}\}.$$

Claim 1. $v^i(+\infty) = w_{i+1}$. Assume by contradiction that $v^i(+\infty) = w_i$. Take $\delta \in]0,\gamma[$ and let $s_1, s_2 \in]-\infty, t^i]$, $s_3, s_4 \in [t^i, +\infty[$ be such that $s_1 < s_2$, $s_3 < s_4$,

$$v^i(s_1), v^i(s_4) \in B_\delta(w_i), \quad v^i(s_2), v^i(s_3) \in \partial B_\gamma(w_i)$$

and

$$\delta \le |v^i(t) - w_i| \le \gamma \ \text{ if } t \in [s_1, s_2] \cup [s_3, s_4].$$

As $v^i_k \to v^i$ uniformly in $[s_1, s_4]$, the same properties hold true for v^i_k when $k \ge k_i$ for some k_i sufficiently large, with a slight modification of δ and γ, say $\tilde{\delta}$ and $\tilde{\gamma}$. Observe that $s_2 - s_1$ and $s_4 - s_3$ are bounded away from zero and as soon as $k \ge k_i$ and $\varepsilon_k < \tilde{\delta}/2$, we have $\omega^i_k < s_1$ and $s_4 < \alpha^{i+1}_k$. It follows that for all $t \in [s_1, s_4]$, $v^i_k(t) = u_k(t)$. Then the argument of Lemma 1.13 and the choice of a sufficiently small δ allow us to modify u_k in $[s_1, s_4]$, joining $u_k(s_1)$ to $u_k(s_4)$ by a segment in such a way that for sufficiently large k, the action of the new function $U_k \in \Gamma(\xi, \eta)$ satisfies the estimate $\mathcal{Q}(U_k) \le c(\xi, \eta) - \rho$ for some $\rho > 0$. This contradicts the definition of the level $c(\xi, \eta)$.

Claim 2. $\mathcal{Q}(v^i) = c(w_i, w_{i+1})$. If the claim is false, there exists some $v \in \Gamma(w_i, w_{i+1})$ with $\mathcal{Q}(v^i) - \mathcal{Q}(v) > 0$. We denote by Δ this last difference. Next, we choose $\delta > 0$, ω, $\alpha \in \mathbb{R}$ such that $\omega < \alpha$,

$$v^i(\omega) \in B_\delta(w_i),\ v^i(\alpha) \in B_\delta(w_{i+1}),\ v^i(t) \ne w_i, w_{i+1} \text{ for all } t \in [\omega, \alpha],$$

$$\mathcal{Q}(v^i) - \mathcal{Q}|^\alpha_\omega(v^i) < \frac{\Delta}{3} \ \text{ and } \ 4\delta^2 + 4 \max_{|z-w_i|\le\delta} V(t,z) < \frac{\Delta}{3}.$$

Let v^i_k denote, as above, the (sub)sequence that converges to v^i. Then $v^i_k(\omega) \in B_\delta(w_i)$, $v^i_k(\alpha) \in B_\delta(w_{i+1})$ for large k and $v^i_k \rightharpoonup v^i$ in $H^1(w, \alpha)$. If u_k first reaches $\partial B_\delta(w_i)$ at time $t = \alpha_k$, then an appropriate translate

$\tilde{u}_k(t) = u_k(t + z_k)$, ($z_k \in \mathbb{Z}$) shifts α_k to $\tilde{\alpha}_k \in [\omega - 2, \omega - 1[$. Similarly, translating the part of u_k after leaving $B_\delta(w_{i+1})$, the escape time β_k with respect to this ball is shifted to $\tilde{\beta}_k \in [\alpha + 1, \alpha + 2[$. Now gluing v to these translated pieces of u_k, we define a new function $U_k \in \Gamma(\xi, \eta)$ by

$$U_k(t) := \begin{cases} \tilde{u}_k(t), & \text{if } t \leq \tilde{\alpha}_k, \\ \dfrac{t - \tilde{\alpha}_k}{\omega - \tilde{\alpha}_k} v(\omega) + \dfrac{\omega - t}{\omega - \tilde{\alpha}_k} \tilde{u}_k(\tilde{\alpha}_k), & \text{if } \tilde{\alpha}_k \leq t \leq \omega, \\ v(t), & \text{if } \omega \leq t \leq \alpha, \\ \dfrac{t - \alpha}{\tilde{\beta}_k - \alpha} \tilde{u}(\tilde{\beta}_k) + \dfrac{\tilde{\beta}_k - t}{\tilde{\beta}_k - \alpha} v(\alpha), & \text{if } \alpha \leq t \leq \tilde{\beta}_k, \\ \tilde{u}_k(t), & \text{if } t \geq \tilde{\beta}_k. \end{cases}$$

A simple computation shows that

$$\begin{aligned} \mathcal{Q}|_{\tilde{\alpha}_k}^{\omega}(U_k) &= \frac{|v(\omega) - \tilde{u}_k(\tilde{\alpha}_k)|^2}{2(\omega - \tilde{\alpha}_k)} + \int_{\tilde{\alpha}_k}^{\omega} V(t, U_k(t))\, dt \\ &\leq 2\delta^2 + 2 \max_{|z - w_i| \leq \delta} V(t, z) \end{aligned}$$

with a similar equality for $\mathcal{Q}|_{\alpha}^{\tilde{\beta}_k}(U_k)$. Noting that

$$\mathcal{Q}(u_k) \geq \mathcal{Q}|_{-\infty}^{\alpha_k}(u_k) + \mathcal{Q}|_{\omega}^{\alpha}(u_k) + \mathcal{Q}|_{\beta_k}^{+\infty}(u_k)$$

and

$$\mathcal{Q}(U_k) \leq \mathcal{Q}|_{-\infty}^{\tilde{\alpha}_k}(\tilde{u}_k) + \mathcal{Q}|_{\omega}^{\alpha}(v) + \mathcal{Q}|_{\tilde{\beta}_k}^{+\infty}(\tilde{u}_k) + 4\delta^2 + 4 \max_{|z - w_i| \leq \delta} V(t, z),$$

where by translation invariance

$$\mathcal{Q}|_{-\infty}^{\tilde{\alpha}_k}(\tilde{u}_k) = \mathcal{Q}|_{-\infty}^{\alpha_k}(u_k) \text{ and } \mathcal{Q}|_{\tilde{\beta}_k}^{+\infty}(\tilde{u}_k) = \mathcal{Q}|_{\beta_k}^{+\infty}(u_k),$$

it turns out that

$$\mathcal{Q}(U_k) \leq \mathcal{Q}(u_k) + \mathcal{Q}|_{\omega}^{\alpha}(v) - \mathcal{Q}|_{\omega}^{\alpha}(u_k) + 4\delta^2 + 4 \max_{|z - w_i| \leq \delta} V(t, z).$$

By our choice of ω, α and δ it follows that

$$\mathcal{Q}(U_k) \leq \mathcal{Q}(u_k) - \frac{\Delta}{3} + \mathcal{Q}|_{\omega}^{\alpha}(v^i) - \mathcal{Q}|_{\omega}^{\alpha}(u_k)$$

and, by weak lower semi-continuity, we conclude that

$$\limsup_{k \to +\infty} \mathcal{Q}(U_k) \leq \lim_{k \to +\infty} \mathcal{Q}(u_k) - \frac{\Delta}{3},$$

which is a contradiction.

Step 3 - Action of the chain. We claim that

$$c(\xi,\eta)=\sum_{i=0}^{j-1}\mathcal{Q}(v^i).$$

Indeed, first note that the weak lower semi-continuity of the norm and Fatou's Lemma yield

$$\sum_{i=0}^{j-1}\mathcal{Q}(v^i)\le\liminf_{k\to+\infty}\sum_{i=0}^{j-1}\mathcal{Q}(v_k^i).$$

On the other hand, a simple computation shows that

$$\sum_{i=0}^{j-1}\mathcal{Q}(v_k^i)\le\mathcal{Q}(u_k)+a_k$$

for some sequence $(a_k)_k$ tending to 0 as k goes to $+\infty$. We therefore deduce that

$$\sum_{i=0}^{j-1}\mathcal{Q}(v^i)\le c(\xi,\eta).\tag{1.29}$$

Assume by contradiction that inequality (1.29) is strict. Choosing $\varepsilon>0$ sufficiently small, we then construct an element $v_\varepsilon\in\Gamma(\xi,\eta)$ by connecting successively each v_i to an appropriate translate of v_{i+1} by a line segment in a suitable neighborhood of w_i, in a such a way that

$$\mathcal{Q}(v_\varepsilon)\le\sum_{i=0}^{j-1}\mathcal{Q}(v^i)+\varepsilon<c(\xi,\eta).$$

This contradicts the definition of $c(\xi,\eta)$. Hence the proof is complete. □

1.5. Notes and Comments

NOTE 1.1. If the solution of the Cauchy problem for (1.12) is unique, any heteroclinic solution of (1.12) connecting -1 to 1 takes values in $]-1,1[$. In addition, if f has only one zero in $]-1,1[$, it can be checked that the minimizers of the functional $\mathcal{J}$ defined by (1.13) are strictly increasing. On the other hand, the presence of the function $a(t)$ in the functional rules out an easier argument, which in the autonomous case shows that the elements of a minimizing sequence may be assumed to be increasing functions.

NOTE 1.2. In comparison with Theorem 1.6, the next theorem illustrates what can be said about second order systems, where the potential depends on time and has no particular symmetry properties. For definiteness, consider the system

$$u'' - \nabla_u V(t,u) = 0, \tag{1.30}$$

where $u : \mathbb{R} \to \mathbb{R}^N$, $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and ∇_u is the gradient with respect to the variable $u \in \mathbb{R}^N$. Then we can prove the following.

THEOREM 1.14. *Assume that $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is such that*
(i) $V(t,u) \geq 0$ for all $t \in \mathbb{R}$ and all $u \in \mathbb{R}^N$, and there exist $\xi \neq \eta$ such that $V(t,u) = 0$ if and only if $u \in \{\xi, \eta\}$;
(ii) there exist constants $a_1, a_2 > 0$ and $\varepsilon > 0$ such that if $z \in \{\xi, \eta\}$ and $|u - z| < \varepsilon$, $a_1|u-z|^2 \leq V(t,u) \leq a_2|u-z|^2$;
(iii) $\liminf_{|u|\to+\infty} V(t,u) > 0$ uniformly in $t \in \mathbb{R}$.
If moreover V satisfies

$$t\frac{\partial V}{\partial t}(t,u) > 0,$$

whenever $t \neq 0$ and $u \notin \{\xi, \eta\}$, then (1.30) *has a heteroclinic solution from ξ to η.*

This theorem is a particular case of the results proved by C.-N. Chen and S.-Y. Tzeng [**32**]. In their paper, further examples of existence and multiplicity results of heteroclinics, may be found. Thus in this respect the non-autonomous equation behaves quite differently from the autonomous one. We also mention that the system considered there may have an infinite set of equilibria.

NOTE 1.3. It is, however, interesting to note that, in contrast to theorems 1.6 and 1.14, P. Korman and A. C. Lazer [**57**] considered a special scalar equation

$$u'' - a(t)(|u|^{p-1}u - u) = 0, \qquad p > 1, \tag{1.31}$$

and they found a heteroclinic from -1 to 1 assuming that $a(t)$ is even, $a'(t) < 0$ if $t < 0$ and $a(+\infty) > 0$. They exploited the symmetry of (1.31), solved the approximate two points boundary value problem with boundary conditions $u(0) = 0$, $u(T) = 1$ and then passed to the limit as $T \to +\infty$.

NOTE 1.4. Many other properties of heteroclinics of Hamiltonian systems have been established by means of the variational methods. Let us list a few of them:

(i) Multi-bump heteroclinics. Consider the system (1.24), where V is a non-negative function of class C^2, T-periodic in t and possessing a global time independent minima at level 0, achieved at exactly two points ξ and η. Then, there exist heteroclinics from ξ to η that travel from a neighborhood of ξ to a neighborhood of η an arbitrary number of times, see P. H. Rabinowitz [**88**].

(ii) Detecting Chaotic Dynamics. The existence of infinitely many families of multi-bump type solutions usually provides evidence of chaotic dynamics. Various authors analyzed the existence of solution with infinitely many bumps (these are no longer multi-bump connecting orbits but limit of sequences of the formers), the conjugation of the flow to a Bernoulli shift and the evaluation of the topological entropy for a large class of Hamiltonian systems. We refer for example to E. Séré [**102, 101**] and E. Bosetto, E. Serra [**19**] for such kind of questions.

(iii) Heteroclinics to periodics. Consider a system of the form

$$u'' - \nabla_u V(t, u) = f(t), \tag{1.32}$$

where V is a function of class C^2, 1-periodic in t and in the space variables, and f is continuous, 1-periodic and of zero mean value. Assume in addition that the system is *reversible* namely that V and f are *even* in t. Then it is known that (1.32) has periodic solutions at the minimum level of the action functional corresponding to the periodic boundary value problem for (1.32). In [**89**], P. H. Rabinowitz gives conditions for the existence of heteroclinics *connecting two such periodic solutions.* The variational formulation for such problems is rather different as one is forced to work with non integrable functions. To remove this obstacle, the main idea of P. H. Rabinowitz was to define a renormalized functional. In [**68**] T. O. Maxwell proved that there are heteroclinic chains between any two periodic solutions, the "vertices" of the chain being also periodic solutions. Theorem 1.12 that we state above is in fact a special, simplified case of Maxwell's result.

M. Calanchi and E. Serra [**28**] used a constraint minimization approach for the existence of connections between consecutive periodic solutions. For ordinary differential equation, E. Bosetto and E. Serra [**19**] obtained multi-bump heteroclinics between consecutive periodic motions without any reversibility assumption.

(iv) Heteroclinics to almost periodic solutions. F. Alessio, C. Carminati and P. Montecchiari [**3**] proved the existence of heteroclinic connections joining almost periodic solutions of a Lagrangian system.

(v) Multi-chain heteroclinics. Suppose that a heteroclinic chain between ξ and η for (1.24) is given. K. H. Strobel [**107**] gave conditions for the existence of a heteroclinic from ξ to η that spends arbitrarily large time in given neighborhoods of the vertices of the heteroclinic chain. Related results may be found in [**91, 94**].

(vi) Heteroclinics with an endpoint at infinity. For a class of potentials $V \in C^1(\mathbb{R}^N, \mathbb{R})$ with one zero at the origin and vanishing at infinity like some power $|u|^{-\alpha}$, $\alpha > 0$, the system (1.24) has at least one "heteroclinic at infinity", i.e. a solution $u(t)$ such that $u(-\infty) = 0$, $|u(+\infty)| = +\infty$ and $u'(\pm\infty) = 0$. We refer to E. Serra [**103**] where a

similar result is also stated for a potential with a finite singularity when $N \geq 3$.

(vii) Almost periodic Lagrangian systems. M. L. Bertotti and P. Montecchiari [**10**] and F. Alessio, M. L. Bertotti and P. Montecchiari [**2**] considered almost periodic Lagrangian systems. Under suitable conditions, they obtained infinitely many heteroclinic solutions connecting possibly degenerate equilibria.

(viii) Singular potentials. Many authors considered homoclinic solutions to equilibrium points of Hamiltonian systems with a singular potential, see [**9, 29, 90, 110**] and the references therein. In case of a potential in $\mathbb{R}^2$ singular at some unique point, the basic result is the existence of many homoclinics classified according to their winding number around the singularity. It seems that only few attention was paid to heteroclinic connections with such settings. A heteroclinic connection from the unique equilibrium to a least energy periodic solution is obtained in P. Caldiroli and L. Jeanjean [**29**] for a conservative singular system. We already mentioned the result of E. Serra [**103**] about heteroclinics at infinity, which holds in some singular frameworks. But up to our knowledge, no multiplicity results were obtained for heteroclinic solutions in the spirit of those concerning homoclinics.

(ix) Heteroclinic connections between minima at different levels of the potential. V. Coti Zelati and P. H. Rabinowitz [**38**] gave conditions for the existence of a heteroclinic from χ to η for a system of the form

$$u'' - a(t)\nabla V(u) = 0,$$

where a is periodic and bounded away from zero and χ, η are isolated minima of V at *different* levels.

(x) Spatial Heteroclinics. Consider the PDE

$$\Delta u - a(t,y)f(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{1.33}$$

where a is periodic in both variables and f is a non-negative smooth function such that $f(0) = f(1) = 0$. There are solutions $u(t,y)$ of (1.33) periodic in y and "heteroclinic in t" from 0 to 1, i.e. approaching respectively 0 and 1 as $t \to -\infty$ and $t \to +\infty$. This and a lot more of related results can be found in P. H. Rabinowitz and E. Stredulinsky [**95**] and P. H. Rabinowitz [**93**].

Note 1.5. In [**7**], M. Arias et al. used variational arguments to deal with travelling waves solutions for equations of the form

$$u'' + cu' + g(u) = 0, \tag{1.34}$$

where $g \in C(]0,1[,\mathbb{R})$ is positive and such that $g(0) = g(1) = 0$. Under suitable conditions on g and the wave speed c, they obtained *fast solutions*, i.e. solutions defined in some interval $[t_0, +\infty]$ and satisfying the integrability condition

$$\int_{t_0}^{+\infty} e^{ct} u'(t)^2 \, dt < +\infty,$$

as minimizers of the functional

$$u \to \int_{\mathbb{R}^+} e^{ct} \left(\frac{u'^2}{2} - G(u) \right) \, dt,$$

where $G(s) := \displaystyle\int_0^s g(\tau) \, d\tau$, defined in the weighted Sobolev space

$$\left\{ u \in H^1_{\mathrm{loc}}(0, +\infty) \mid \int_{\mathbb{R}^+} e^{ct} u'(t)^2 \, dt < +\infty \text{ and } u(+\infty) = 0 \right\}.$$

Using a classical reduction to a first order equation, see e.g. [**98**], they derived subsequent existence results for heteroclinic solutions of (1.34) connecting 1 to 0.

Chapter 2

Minimization of Positive Functionals

From this chapter, we focus on fourth order bi-stable differential equation. We first consider the model equation

$$u'''' - \beta u'' + u^3 - u = 0,$$

with $\beta \geq 0$, which was extensively studied, for instance, by L. A. Peletier and W. C. Troy [**82**], L. A. Peletier, W. C. Troy and R. C. A. M. VanderVorst [**77**], W. D. Kalies and R. C. A. M. VanderVorst [**55**] and W. D. Kalies, J. Kwapisz and R. C. A. M. VanderVorst [**54**]. Though we later deal with more general equations, it is worth considering this simple model equation to single out the main arguments and make a distinction between further difficulties. Formally, the above model equation is the Euler-Lagrange equation corresponding to the functional

$$\mathcal{F}_\beta(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2-1)^2 \right) dx.$$

In Section 2.1, we show that this last functional does achieve a minimum in an appropriate space of functions and any minimizer is a heteroclinic solution connecting -1 to $+1$.

We then proceed to a broader framework. It is meaningful, in some applications [**48**], to replace the parameter β by a non-constant term $g(u)$ in the functional $\mathcal{F}_\beta$. We may also substitute the model potential $(u^2-1)^2/4$ by any double-well non-negative potential f. This leads to the following functional

$$\mathcal{F}_g(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx.$$

The non-constant term $g(u)$ in the functional does not bring new difficulties as long as g is a non-negative function. On the other hand, the problem is quite different when g changes sign as it is no longer clear whether the functional is bounded from below or no. We tackle this question in Chapter 3. In Section 2.2, assuming that g is a non-negative function and f is a non-negative double-well potential, we prove the existence of a minimizer of the functional $\mathcal{F}_g$ under quite weak nondegeneracy conditions at the bottoms of the potential. The minimizer is then a solution of

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0$$

connecting the minima of f.

The results we present are particular cases of those in D. Bonheure and L. Sanchez [**17**]. These were originally published in P. Habets, L. Sanchez, M. Tarallo and S. Terracini [**49**] and generalized in D. Bonheure, L. Sanchez, M. Tarallo and S. Terracini [**18**].

In the last section of this chapter, we investigate the qualitative properties of the heteroclinic solutions obtained by minimization. We recall the arguments introduced by W. D. Kalies et al. [**54**] and present a slight simplification of their clipping method as worked out in D. Bonheure and L. Sanchez [**17**]. We then apply their arguments to derive some information concerning the shape of the minimizers. The study of the way the minimizers converge to the equilibria ± 1 at $\pm\infty$ requires non-variational arguments. W. D. Kalies et al. obtained a rather precise description of the behaviour of the minimizers at $\pm\infty$ when the equilibria are saddle-foci. Here, we follow an alternative approach that was presented in D. Bonheure, P. Habets and L. Sanchez [**16**].

2.1. The Extended Fisher-Kolmogorov Equation

As already mentioned in the introduction, heteroclinic solutions of the Extended Fisher-Kolmogorov equation

$$u'''' - \beta u'' + u^3 - u = 0 \tag{2.1}$$

are critical points of the action functional

$$\mathcal{F}_\beta(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2-1)^2 \right) dx. \tag{2.2}$$

This functional is well defined in the space

$$\mathcal{E} = \{u : \mathbb{R} \to \mathbb{R} \mid u(0) = 0,\ u+1 \in H^2(\mathbb{R}^-),\ u-1 \in H^2(\mathbb{R}^+)\}. \tag{2.3}$$

The condition $u(0) = 0$ is convenient to avoid a loss of compactness due to the invariance under translations. Clearly, it is in no way restrictive as any translate of a solution is still a solution.

Definition 2.1. *A heteroclinic solution of equation* (2.1) *connecting* -1 *to* $+1$ *is a solution* $u \in C^4(\mathbb{R})$ *of* (2.1) *that satisfies*

$$\lim_{x\to\pm\infty} (u(x), u'(x), u''(x), u'''(x)) = (\pm 1, 0, 0, 0).$$

PROPOSITION 2.2. *Let $\beta \in \mathbb{R}$. The functional $\mathcal{F}_\beta : \mathcal{E} \to \mathbb{R}$ defined by* (2.2) *and* (2.3) *is of class C^1 and any critical point is a C^∞ heteroclinic solution of* (2.1) *connecting -1 to $+1$.*

PROOF. We claim that the Fréchet derivative of $\mathcal{F}_\beta$ at point $u \in \mathcal{E}$ is given by $\mathcal{F}'_\beta(u) : H^2(\mathbb{R}) \to \mathbb{R}$, where

$$\mathcal{F}'_\beta(u)\varphi := \int_{\mathbb{R}} \left(u''\varphi'' + \beta u'\varphi' + (u^3 - u)\varphi\right) dx.$$

Indeed, writing $f(u) = u^3 - u$ and $F(u) = \dfrac{(u^2-1)^2}{4}$, we have

$$\left| \int_{\mathbb{R}} \left(\frac{1}{2}\left(\varphi''^2 + \beta\varphi'^2\right) + F(u+\varphi) - F(u) - f(u)\varphi \right) dx \right| \leq C\|\varphi\|^2_{H^2(\mathbb{R})},$$

where we used the continuous injection of $H^2(\mathbb{R})$ in $C(\mathbb{R})$. We then infer that

$$\mathcal{F}_\beta(u+\varphi) = \mathcal{F}_\beta(u) + \mathcal{F}'_\beta(u)\varphi + o(\|\varphi\|_{H^2(\mathbb{R})})$$

as $\|\varphi\|_{H^2(\mathbb{R})} \to 0$.

Now let u be a critical point of $\mathcal{F}_\beta$. We then have $\mathcal{F}'_\beta(u)\varphi = 0$ for every $\varphi \in H^2(\mathbb{R})$ satisfying $\varphi(0) = 0$. Starting with $\varphi \in C^2_c(\mathbb{R})$ and using Du Bois-Reymond Lemma, we easily find that in fact $u \in C^4(\mathbb{R})$ and u solves (2.1). As this equation expresses u'''' in terms of u'' and $u-1$, we deduce that $u'''' \in L^2(\mathbb{R}^+)$. Indeed, $u'' \in L^2(\mathbb{R}^+)$ and

$$\int_{\mathbb{R}^+} (u^3(x) - u(x))^2\, dx \leq (\|u\|^2_\infty + \|u\|_\infty)^2 \|u-1\|^2_{L^2(\mathbb{R}^+)} < +\infty.$$

It now follows by interpolation that

$$\|u'''\|^2_{L^2(\mathbb{R}^+)} \leq C(\|u''\|^2_{L^2(\mathbb{R}^+)} + \|u''''\|^2_{L^2(\mathbb{R}^+)}) < +\infty.$$

Hence $u-1 \in H^4(\mathbb{R}^+)$. We prove in the same way that $u+1 \in H^4(\mathbb{R}^-)$. As well-known, the L^2-integrability of the derivatives implies that

$$\lim_{x\to\pm\infty} u(x) = \pm 1 \quad \text{and} \quad \lim_{x\to\pm\infty} u^{(n)}(x) = 0 \quad \text{for } n = 1, 2, 3$$

so that u is a heteroclinic solution of (2.1) connecting -1 to $+1$. Finally, a straightforward bootstrap argument shows that indeed u is of class C^∞. □

For $\beta \geq 0$, $\mathcal{F}_\beta$ is positive and hence bounded from below. The a priori simplest way to find critical points of $\mathcal{F}_\beta$ is therefore to search for minimizers. The next theorem confirms the efficiency of a minimization approach.

THEOREM 2.3. *For all $\beta \geq 0$, the functional $\mathcal{F}_\beta : \mathcal{E} \to \mathbb{R}$ defined by* (2.2) *and* (2.3) *has a minimizer which is a heteroclinic solution of* (2.1) *connecting -1 to $+1$. Furthermore, any minimizer is odd and positive in $]0, +\infty[$.*

This theorem is the analogous of Theorem 1.2. Remember that for the functional $\mathcal{I}$ of Theorem 1.2, minimizing sequences $(u_n)_n$ can be taken in such a way that $-1 \leq u_n \leq 1$. Indeed, $v_n = \sup(-1, \inf(u_n, 1))$ is still a minimizing sequence. Using similar modification arguments it is easily seen that the minimizers of $\mathcal{I}$ are monotone.

One of the first differences that we encounter when considering the functional $\mathcal{F}_\beta$ in $\mathcal{E}$, comes from the fact that these modifications which keep functions in $H^1_{\mathrm{loc}}(\mathbb{R})$ do not necessarily produce functions of class C^1 and therefore, in general, the modified functions do not belong to $H^2_{\mathrm{loc}}(\mathbb{R})$. By the way, as already mentioned, it is not true in general that minimizers of $\mathcal{F}_\beta$ in $\mathcal{E}$ are monotone. When dealing with $\mathcal{I}$, modification arguments are also used to ensure that quasi-minimizers stay close to ± 1 outside a fixed compact interval which can always be centered around zero by translation invariance.

To substitute these rather simple arguments, we make the most of the symmetry of the functional $\mathcal{F}_\beta$ and we exploit the quadraticity of the potential around its minima. The original proof of Theorem 2.3 is due to L. A. Peletier, W. C. Troy and R. C. A. M. VanderVorst [**77**]. We give here a proof which is closer to W. D. Kalies and R. C. A. M. VanderVorst [**55**].

Proof of Theorem 2.3. We introduce the spaces

$$\mathcal{E}^+ = \{u : \mathbb{R}^+ \to \mathbb{R} \mid u(0) = 0,\ u - 1 \in H^2(\mathbb{R}^+)\},$$

$$\mathcal{E}^- = \{u : \mathbb{R}^- \to \mathbb{R} \mid u(0) = 0,\ u + 1 \in H^2(\mathbb{R}^-)\}$$

and consider the restricted functional $\mathcal{F}^\pm_\beta : \mathcal{E}^\pm \to \mathbb{R}$ by

$$\mathcal{F}^\pm_\beta(u) = \int_{\mathbb{R}^\pm} L_\beta(u, u', u'')\, dx,$$

where $L_\beta(u, u', u'')$ is the Lagrangian given by

$$L_\beta(u, u', u'') = \frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2 - 1)^2. \tag{2.4}$$

Let us define the values

$$c := \inf_{\mathcal{E}} \mathcal{F}_\beta \text{ and } c^\pm := \inf_{\mathcal{E}^\pm} \mathcal{F}^\pm_\beta.$$

Since $\mathcal{F}_\beta$ is symmetric, it is easily seen that for all $u^+ \in \mathcal{E}^+$,

$$\mathcal{F}^+_\beta(u^+) = \mathcal{F}^-_\beta(u^-),$$

where $u^- \in \mathcal{E}^-$ is defined by $u^-(x) = -u^+(-x)$. Therefore, we deduce that

$$c^+ = c^- = \frac{c}{2}.$$

Claim 1. The variational problem

$$\inf\{\mathcal{F}_\beta^+(u) \mid u \in \mathcal{E}^+\}$$

has a positive solution.

Let $(v_n)_n \subset \mathcal{E}^+$ be a minimizing sequence for $\mathcal{F}_\beta^+$, i.e. $v_n \in \mathcal{E}^+$ for all $n \in \mathbb{N}$ and $\mathcal{F}_\beta^+(v_n) \to c^+$. For each $n \geq 0$, we define

$$x_n := \sup\{x \geq 0 \mid v_n(x) = 0\}.$$

Since $\lim_{x\to+\infty} v_n(x) = 1$, $x_n < +\infty$ for all $n \geq 0$. We now consider the positive sequence $(v_n^+)_n \subset \mathcal{E}^+$, where

$$v_n^+(x) := v_n(x + x_n)$$

for $x \geq 0$. We observe that

$$\int_0^{x_n} L_\beta(v_n, v_n', v_n'')\, dx \geq 0$$

so that $\mathcal{F}_\beta^+(v_n^+) \leq \mathcal{F}_\beta^+(v_n)$ which implies that $(v_n^+)_n$ is also a minimizing sequence for $\mathcal{F}_\beta^+$.

As the sequence $\mathcal{F}_\beta^+(v_n^+)$ is uniformly bounded, we deduce a uniform estimate for $\|v_n^+ - 1\|_{H^2(\mathbb{R}^+)}$. Indeed, the L^2-bounds for $(v_n^+)'$ and $(v_n^+)''$ follow easily from the bound on $\mathcal{F}_\beta^+(v_n^+)$, while we infer from the positivity of v_n^+ that

$$\int_{\mathbb{R}^+} \frac{(v_n^+ - 1)^2}{4}\, dx \leq \int_{\mathbb{R}^+} \frac{(v_n^{+2} - 1)^2}{4}\, dx \leq \mathcal{F}_\beta^+(v_n^+).$$

We now deduce that going to a subsequence if necessary, there exists

$$v^+ \in H^2(\mathbb{R}^+) + 1$$

such that

$$v_n^+ - 1 \overset{H^2(\mathbb{R}^+)}{\rightharpoonup} v^+ - 1$$

and

$$v_n^+ \xrightarrow{C^1_{\mathrm{loc}}(\mathbb{R}^+)} v^+.$$

As the two first terms in $\mathcal{F}_\beta^+$ are the square of seminorms and Fatou's Lemma is applicable to the last one, we deduce that

$$\mathcal{F}_\beta^+(v^+) \leq \liminf_{n\to+\infty} \mathcal{F}_\beta^+(v_n^+) = c^+.$$

The convergence being uniform on compact intervals, we conclude that $v^+(0) = 0$ so that $v^+ \in \mathcal{E}^+$ and $\mathcal{F}_\beta^+(v^+) = c^+$. Observe that v^+ is positive on $]0, +\infty[$ otherwise we could proceed as above to construct a positive function having smaller action.

Claim 2. If $v \in \mathcal{E}^+$ is such that $(\mathcal{F}_\beta^+)'(v) = 0$, then $v''(0) = 0$, $v^* \in \mathcal{E}$ defined by

$$v^*(x) := \begin{cases} v(x) & \text{if } x \geq 0, \\ -v(-x) & \text{if } x < 0, \end{cases} \tag{2.5}$$

is a minimizer of $\mathcal{F}_\beta$ in $\mathcal{E}$ and v^* is a heteroclinic solution of (2.1).

We first compute

$$(\mathcal{F}_\beta^+)'(v)(\varphi) = \int_{\mathbb{R}^+} \left(v''\varphi'' + \beta v'\varphi' + (v^3 - v)\varphi\right) dx \tag{2.6}$$

for all $\varphi \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$. Arguing as in Proposition 2.2, we deduce that v solves equation (2.1), $v \in C^4(\mathbb{R}^+)$ and

$$\lim_{x\to+\infty} v(x) = +1 \quad \text{and} \quad \lim_{x\to+\infty} v^{(n)}(x) = 0 \quad \text{for } n = 1, 2, 3. \tag{2.7}$$

Integrating (2.6) by parts, we obtain for all $\varphi \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$,

$$[v''\varphi']_0^{+\infty} - [v'''\varphi]_0^{+\infty} - \beta[v'\varphi]_0^{+\infty} = 0$$

as

$$\int_{\mathbb{R}^+} (v'''' - \beta v'' + v^3 - v)(x)\varphi(x)\, dx = 0.$$

Taking (2.7) into account, we now deduce that

$$v''(0)\varphi'(0) = 0$$

for all $\varphi \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$ which obviously implies that $v''(0) = 0$. The function $v^* : \mathbb{R} \to \mathbb{R}$ defined by (2.5) is therefore of class C^4, solves (2.1) and as

$$\mathcal{F}_\beta(v^*) = 2\mathcal{F}_\beta^+(v) = 2c^+ = c,$$

we conclude that v^* is a minimizer of $\mathcal{F}_\beta$ in $\mathcal{E}$.

Claim 3. If $u \in \mathcal{E}$ minimizes $\mathcal{F}_\beta$, then u is odd.

Let us define $u^\pm := u_{|\mathbb{R}^\pm}$. As $\mathcal{F}_\beta(u) = c$, we obviously have

$$\mathcal{F}_\beta^+(u^+) = \mathcal{F}_\beta^-(u^-) = \frac{c}{2}$$

otherwise the odd extension of u^+ or u^- would have a lower action than c. Define $v^+ \in \mathcal{E}^+$ by $v^+(x) = -u^-(-x)$. Then, v^+ satisfies $\mathcal{F}_\beta^+(v^+) = c^+$ and therefore minimizes $\mathcal{F}_\beta^+$ in $\mathcal{E}^+$. It follows that both u^+ and v^+ are minimizers of $\mathcal{F}_\beta^+$ in $\mathcal{E}^+$. From Claim 2, we next infer that $u''(0) = (v^+)''(0) = 0$ and as $(v^+)'(0) = (u^-)'(0) = (u^+)'(0)$ and $(v^+)'''(0) = (u^-)'''(0) = (u^+)'''(0)$, the functions v^+ and u^+ solve the same Cauchy problem

$$\begin{aligned} u''''(x) - \beta u''(x) + u^3(x) - u(x) = 0,\ \ x \geq 0 \\ u(0) = 0,\ u'(0) = (u^+)'(0), \\ u''(0) = 0,\ u'''(0) = (u^+)'''(0). \end{aligned}$$

By uniqueness, this implies $u^+(x) = v^+(x)$ for all $x \in \mathbb{R}^+$, that is $u^+(x) = -u^-(-x)$ for all $x \in \mathbb{R}^+$. $\square$

2.2. Double-well Potentials with Degenerate Minima

The arguments of the preceding section easily extend to a functional of the form

$$\mathcal{F}_g(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx, \tag{2.8}$$

where g is a positive even function and f is a symmetric potential having two nondegenerate minima at the same energy level and superquadratic grows at $\pm\infty$. The corresponding Euler-Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{2.9}$$

In this section, we drop the symmetry assumption on the Lagrangian

$$L_g(u, u', u'') = \frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \tag{2.10}$$

and we enlarge the class of potentials. The main assumptions are the following. We consider a double well potential $f \in C^1(\mathbb{R})$ such that

(B1) $f(u) = 0$ if and only if $u = \pm 1$;

(B2) for some $0 < a < 1$ and $\alpha > 0$,

$$\frac{f(u)}{(u-1)^2} \leq \alpha, \text{ for } |u-1| < a,$$
$$\frac{f(u)}{(u+1)^2} \leq \alpha, \text{ for } |u+1| < a;$$

(B3) $f(u) \geq 0$ for all $u \in \mathbb{R}$ and

$$\liminf_{|u| \to +\infty} f(u) > 0.$$

We also assume that $g \in C^1(\mathbb{R})$ is a non-negative function. Notice that assumption (B2) is automatically satisfied when f is twice differentiable around ± 1.

Under those assumptions, it is not true in general that a function u whose action is finite is such that $u - 1$ and $u + 1$ are square-integrable on respectively $\mathbb{R}^+$ and $\mathbb{R}^-$. Indeed, the potential could be too "flat" around ± 1. Hence it makes no sense to consider $\mathcal{F}_g$ in the functional space $\mathcal{E}$. In fact it is sufficient to search for a minimizer of $\mathcal{F}_g$ in the set

$$\mathcal{H} = \{u \in C^1(\mathbb{R}) \mid u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}),\ \lim_{x \to \pm\infty} u(x) = \pm 1\}. \tag{2.11}$$

On the other hand, a minimizer in $\mathcal{H}$ is only a heteroclinic solution in a weak sense as we cannot prove that all derivatives up to the third order vanish at $\pm\infty$. We therefore introduce the following definition.

DEFINITION 2.4. *Let $g \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R})$ be a non-negative potential that vanishes at ± 1. A solution $u \in C^4(\mathbb{R})$ of* (2.9) *is a weak heteroclinic solution connecting -1 to $+1$ if u satisfies*

$$\lim_{x \to \pm\infty} u(x) = \pm 1$$

and

$$H(u,\, u',\, u'',\, u''') := u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u) = 0. \tag{2.12}$$

The left term in equation (2.12) is nothing but the *Hamiltonian* corresponding to equation (2.9). In other words, a solution connecting -1 to $+1$ is a weak heteroclinic solution if it lies in the energy manifold through the equilibria ± 1.

It is obvious that a classical heteroclinic solution of (2.9) is a weak heteroclinic. Indeed, as the equation is conservative, H is a first integral which means that for any solution u, there exists a constant E such that for every $x \in \mathbb{R}$,

$$H(u(x),\, u'(x),\, u''(x),\, u'''(x)) = E.$$

When u is a classical heteroclinic connection between -1 and $+1$, it is easily seen that $E = 0$.

Our first observation is that the first derivative of any function $u \in \mathcal{H}$ that satisfies $\mathcal{F}_g(u) < +\infty$, vanishes at $\pm\infty$. To prove this, we state the following useful estimate.

LEMMA 2.5. *Given an interval $[a, b] \subset \mathbb{R}$ and a function $u \in H^2(a, b)$ such that $u(a) = A$, $u(b) = B$, $u'(a) = A_1$, $u'(b) = B_1$, the following inequality holds*

$$\int_a^b u''^2\, dx \geq \frac{4}{b-a}\left((B_1 - A_1)^2 + 3\left(\frac{B-A}{b-a} - A_1\right)\left(\frac{B-A}{b-a} - B_1\right)\right),$$

and equality holds if and only if u is a third degree polynomial.

PROOF. Denote by P the third degree polynomial such that P and P' coincide respectively with u and u' at points a and b. Writing u as $P + w$, we compute

$$\int_a^b u''^2\, dx = \int_a^b P''^2\, dx + \int_a^b w''^2\, dx + 2\int_a^b P''w''\, dx. \tag{2.13}$$

Integrating $P''w''$ by parts and using the fact that $w(a) = w(b) = 0$ and $w'(a) = w'(b) = 0$, we see that the last integral in the right-hand side of

(2.13) is actually zero. We thus obtain the inequality

$$\int_a^b u''^2\,dx \geq \int_a^b P''^2\,dx$$

and the conclusion now follows by computing the integral of P''^2. □

REMARK 1.1. Another way to prove Lemma 2.5 is to consider the minimization problem

$$\min\left(\int_a^b u''^2\,dx\right)$$

in the set of functions $u \in H^2(a,b)$ such that $u(a) = A$, $u(b) = B$, $u'(a) = A_1$, $u'(b) = B_1$. It is clear that the minimum is achieved by a solution of the fourth order equation

$$u'''' = 0.$$

Taking the boundary conditions into account, we deduce that the unique minimizer is the third degree polynomial satisfying those conditions. The desired estimate then easily follows.

LEMMA 2.6. *Assume that f satisfies* (B1) *and g is a non-negative function. Let $u \in \mathcal{H}$ be such that $\mathcal{F}_g(u) < +\infty$. Then*

$$\lim_{x\to\pm\infty} u'(x) = 0.$$

PROOF. Let $\varepsilon > 0$ and $u \in \mathcal{H}$ be given. Suppose by contradiction that our conclusion is false. Assume for example that $u'(x_n) \geq \varepsilon$ for some sequence $x_n \to +\infty$, $n \in \mathbb{N}$. Let $\delta > 0$ be such that

$$\delta < \frac{\varepsilon^3}{16\mathcal{F}(u)}.$$

Then, as $u(+\infty) = 1$, there exists $R > 0$ such that for all $x \geq R$,

$$|u(x) - 1| \leq \frac{\delta}{2}.$$

Let $x_0 > R$ be such that $u'(x_0) = \varepsilon$. We claim that we can find $x_1 > 0$ such that

$$u'(x_1) = \frac{\varepsilon}{2},\ u(x_1) \leq u(x_0) + \delta \text{ and } x_1 - x_0 \leq \frac{2\delta}{\varepsilon}.$$

Indeed, if $x > x_0$, we have $|u(x) - u(x_0)| \leq \delta$ and as $u(+\infty) = 1$, there exists $x > x_0$ such that $u'(x) \leq \varepsilon/2$. We can therefore choose x_1 such that $u'(x) \geq \varepsilon/2$ for all $x \in [x_0, x_1]$. We then have

$$\frac{\varepsilon}{2}(x_1 - x_0) \leq \int_{x_0}^{x_1} u'(s)\,ds = u(x_1) - u(x_0) \leq \delta.$$

Now, letting $m = \dfrac{u(x_1) - u(x_0)}{x_1 - x_0}$, we infer from Lemma 2.5 that

$$\int_{x_0}^{x_1} u''^2\, dx \geq \frac{2\varepsilon}{\delta}\left(\left(\frac{\varepsilon}{2}\right)^2 + 3(m-\varepsilon)(m-\frac{\varepsilon}{2})\right) \geq \frac{\varepsilon^3}{8\delta}.$$

Hence, we obtain a contradiction with the choice of δ since

$$2\mathcal{F}(u) \geq \int_{x_0}^{x_1} u''^2\, dx \geq \frac{\varepsilon^3}{8\delta}.$$

Similar arguments hold in the other cases, in particular if we have $u'(-\infty) \neq 0$. □

With these preliminaries, we can prove that a minimizer in $\mathcal{H}$ is a weak heteroclinic solution connecting -1 to $+1$.

PROPOSITION 2.7. *Let f satisfy* (B1) *and $g \in C^1(\mathbb{R})$ be non-negative. If $u \in \mathcal{H}$ minimizes the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by* (2.8) *and* (2.11)*, then u is a weak heteroclinic solution of* (2.9) *connecting -1 to $+1$ and fulfilling $u'(\pm\infty) = 0$.*

PROOF. Let $u \in \mathcal{H}$ be a minimizer of $\mathcal{F}_g$. Since the action of u is finite, Lemma 2.6 implies that u' vanishes at both $\pm\infty$. The fact that u solves the Euler-Lagrange equation follows from by now familiar arguments. To prove that u is a weak heteroclinic solution, we argue by contradiction. Assume that

$$H(u, u', u'', u''') \neq 0,$$

where H is as in (2.12). The conservation of the Hamiltonian then implies that for some $E \in \mathbb{R}_0$,

$$H(u, u', u'', u''') = E.$$

Let $(x_n)_n \subset \mathbb{R}^+$ be such that $x_n \to +\infty$ and $(u''(x_n))_n \subset \mathbb{R}$ is a bounded sequence. Since $u'(+\infty) = 0$, such a sequence obviously exists. Then we have

$$\int_0^{x_n} (u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u))\, dx = Ex_n.$$

Integrating by parts the first term on the left hand-side, we notice that the integral is bounded contradicting the equality. □

We now turn to the search of a minimizer.

THEOREM 2.8. *Suppose that $g \in C^1(\mathbb{R})$ is a non-negative function and $f \in C^1(\mathbb{R})$ satisfies assumptions* (B1)*,* (B2) *and* (B3)*. Then, the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by* (2.8) *and* (2.11) *achieves a minimum. A minimizer is a weak heteroclinic solution of* (2.9) *connecting -1 to $+1$.*

To prove that $\mathcal{F}_g$ achieves a minimum in $\mathcal{H}$, we need sharp estimates on a minimizing sequence. The key arguments are summarized in the next proposition.

PROPOSITION 2.9. *Suppose that $g \in C^1(\mathbb{R})$ is a non-negative function and $f \in C^1(\mathbb{R})$ satisfies* (B1), (B2) *and* (B3). *Then there exist $L > 0$, $T > 0$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $\mathcal{F}(u_n) \to \inf_{\mathcal{H}} \mathcal{F}$ and for all $n \in \mathbb{N}$,*
(i) $\|u_n\|_{C^1} \leq L$,
(ii) $|u_n(x) + 1| \leq a$ *for all* $x \leq -T$ *and* $|u_n(x) - 1| \leq a$ *for all* $x \geq T$.

Observe that the first property of the minimizing sequence $(u_n)_n$ allows to choose a converging subsequence in C^1_{loc}. The second property then permit to overcome the lack of compactness of the functional due to its invariance under translations in the x-variable. We postponed the proof of Proposition 2.9 to Section 2.2.1. On the other hand, the proof of Theorem 2.8 now easily follows from the existence of a minimizing sequence satisfying properties (i) and (ii).

PROOF OF THEOREM 2.8. Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence satisfying the conclusions of Proposition 2.9.

Step 1 - Convergence. Since each term of the Lagrangian $L_g(u, u', u'')$ is non-negative, the sequence $(u''_n)_n$ is bounded in $L^2(\mathbb{R})$. Together with the uniform bound on u_n in $C^1(\mathbb{R})$ (property (i) of the minimizing sequence), this implies that $(u_n)_n$ has a subsequence (still written $(u_n)_n$ for simplicity) such that for some function $u \in H^2_{\text{loc}}(\mathbb{R})$,

$$u_n \overset{C^1_{\text{loc}}(\mathbb{R})}{\longrightarrow} u, \quad u''_n \overset{L^2(\mathbb{R})}{\rightharpoonup} u''.$$

We also infer from Fatou's Lemma that

$$\int_{\mathbb{R}} f(u(x))\, dx < +\infty. \tag{2.14}$$

Step 2 - $u \in \mathcal{H}$. Observe first that $u' \in L^\infty(\mathbb{R})$. Indeed, as u'_n converges uniformly to u' on every compact subset of $\mathbb{R}$, it follows from the uniform bound on $\|u'_n\|_\infty$ that u' is bounded in $\mathbb{R}$.

We next prove that

$$\lim_{x\to+\infty} u(x) = 1.$$

From the convergence on compact sets, it is clear that $|u(x) - 1| \leq a$ for $x \geq T$. We therefore deduce that

$$1 - a \leq \liminf_{x\to+\infty} u(x) \leq \limsup_{x\to+\infty} u(x) \leq 1 + a.$$

Assume by contradiction that $\limsup_{x\to+\infty} u(x) > 1$. Then, we obviously deduce from (2.14) that

$$\liminf_{x\to+\infty} u(x) \leq 1.$$

This means that for some $0 < \varepsilon < a$, there exist infinitely many disjoint intervals $[a_i, b_i] \subset \mathbb{R}^+$, $i \in \mathbb{N}$ such that

$$1 + \frac{\varepsilon}{2} \leq u(x) \leq 1 + \varepsilon \text{ for all } x \in [a_i, b_i].$$

We can suppose without loss of generality that $u(b_i) - u(a_i) = \pm\frac{\varepsilon}{2}$. As $\|u'\|_\infty \leq L$, we infer that

$$\frac{\varepsilon}{2} = |u(b_i) - u(a_i)| = \left| \int_{a_i}^{b_i} u'(s)\, ds \right| \leq L(b_i - a_i)$$

which implies that

$$(b_i - a_i) \geq \frac{\varepsilon}{2L}.$$

Now, let $m_\varepsilon > 0$ be such that $f(u) \geq m_\varepsilon$ for all $u \in [1 + \frac{\varepsilon}{2}, 1 + \varepsilon]$. We then compute

$$\int_{\mathbb{R}^+} f(u(x))\, dx \geq \sum_{i=0}^{\infty} \int_{a_i}^{b_i} f(u(x))\, dx \geq \sum_{i=0}^{\infty} \frac{\varepsilon\, m_\varepsilon}{2L} = +\infty$$

which contradicts (2.14). If $\liminf_{x\to+\infty} u(x) < 1$, we derive the same contradiction so that

$$\liminf_{x\to+\infty} u(x) = \limsup_{x\to+\infty} u(x) = 1.$$

As we can argue similarly if

$$\limsup_{x\to-\infty} u(x) > -1 \quad \text{or} \quad \liminf_{x\to-\infty} u(x) < -1,$$

we also deduce that $u(-\infty) = -1$. Summing up, we come to the conclusion that $u \in \mathcal{H}$.

Conclusion - To see that u is a minimizer of $\mathcal{F}_g$, it suffices to observe that the first term in $\mathcal{F}_g$ is the square of a seminorm and Fatou's Lemma is applicable to the last two. Hence we deduce that

$$\mathcal{F}_g(u) \leq \liminf_{n\to+\infty} \mathcal{F}_g(u_n) = \inf_{\mathcal{H}} \mathcal{F}$$

which by Step 2 implies

$$\mathcal{F}_g(u) = \inf_{\mathcal{H}} \mathcal{F}_g.$$

The last statement of the theorem follows from Proposition 2.7. $\square$

2.2.1. Proof of Proposition 2.9. We devote the remaining of the section to the proof of Proposition 2.9. We divide the proof in two parts. In the first part we prove the a priori bound (property (i)) and in the second one we prove the localization of at least one minimizing sequence (property (ii)). For property (ii), the key argument is that given two neighborhoods N_{-1} of $(-1, 0)$ and N_{+1} of $(+1, 0)$ in the (u, u')-plane, we can estimate the time that a function of $\mathcal{H}$ takes to travel from N_{-1} to N_{+1}. We then approximate the minimum of $\mathcal{F}_g$ with functions that stay close to ± 1 at $\pm\infty$. These arguments have to be compared with Lemma 1.4 and Lemma 1.5 in Section 1.2.2.

As in Chapter 2, for a functional $\mathcal{F}$ defined from a Lagrangian $L(u, u', u'')$ and $S \subset \mathbb{R}$, we extensively use the notation

$$\mathcal{F}|_S(u) = \int_S L(u, u', u'')\, dx$$

and when S is an interval, say $[a, b]$, we write $\mathcal{F}|_a^b(u)$.

Proof of Proposition 2.9. Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence of $\mathcal{F}_g$. As $\mathcal{F}_g(u_n)$ is a converging sequence, $\mathcal{F}_g(u_n)$ is uniformly bounded, say by some positive constant C.

Part 1 - There exists $L > 0$ such that for all $n \in \mathbb{N}$, $\|u_n\|_{C^1} \leq L$.

According to the assumptions on f there exist $K > 0$ and $b > 0$ such that

$$\frac{3K^2}{8b^3} > C \quad \text{and} \quad f(u) > \frac{C}{b} \quad \text{for all } |u| \geq \frac{K}{2}.$$

Claim 1. We claim that $\|u_n\|_{L^\infty(\mathbb{R})} \leq K$. Otherwise, either the set

$$\{x \in \mathbb{R} \mid \; |u_n(x)| \geq K/2\}$$

has measure greater than b and

$$\mathcal{F}_g(u_n) \geq \int_{\mathbb{R}} f(u_n)\, dx > C,$$

a contradiction, or we can pick up an interval $[c, d]$ such that $d - c < b$, $|u_n(c)| = \|u_n\|_{L^\infty(\mathbb{R})}$, $|u_n(d)| = \frac{K}{2}$, $|u_n(x)| \geq \frac{K}{2}$, for all $x \in [c, d]$. It then follows, using Lemma 2.5, that

$$\begin{aligned}
\mathcal{F}_g(u_n) &\geq \int_c^d \frac{u_n''^2}{2}\, dx \\
&\geq \frac{2}{d-c}\left(u_n'(d)^2 + 3\bigg(\frac{u_n(d)-u_n(c)}{d-c}\bigg)\bigg(\frac{u_n(d)-u_n(c)}{d-c} - u_n'(d)\bigg)\right) \\
&\geq \frac{3}{2(d-c)}\left(\frac{u_n(d)-u_n(c)}{d-c}\right)^2 \\
&\geq \frac{3K^2}{8b^3} > C,
\end{aligned}$$

leading again to a contradiction. Hence the bound on $\|u_n\|_{L^\infty(\mathbb{R})}$ is established.

Claim 2. There exists $M > 0$ such that $\|u'\|_{L^\infty(\mathbb{R})} \leq M$. We may choose $M > 0$ in such a way that

$$M > 4K \quad \text{and} \quad M^2 > 8C.$$

Indeed, assume by contradiction that $u_n'(x_0) = M$. Then, there exists $x_1 \in [x_0, x_0+1]$ such that $u_n'(x_1) = \frac{M}{2}$. Hence, denoting again

$$m = \frac{u_n(x_1) - u_n(x_0)}{x_1 - x_0},$$

it turns out that

$$\begin{aligned}
\mathcal{F}_g(u_n) &\geq \int_{x_0}^{x_1} \frac{u_n''^2}{2}\, dx \\
&\geq \frac{2}{x_1 - x_0}\left(\left(\frac{M}{2}\right)^2 + 3\,(m - M)\left(m - \frac{M}{2}\right)\right) \\
&\geq \frac{M^2}{8} > C,
\end{aligned}$$

which is impossible. We show in a similar way that u_n' cannot attain the value $-M$.

We thus deduce from the previous claims that the conclusion of Part 1 holds with

$$L = \max(K, M).$$

PART 2 - There exists $T > 0$ and a minimizing sequence $(v_n)_n \subset \mathcal{H}$ such that for all $n \in \mathbb{N}$,

$$|v_n(x) + 1| \leq a \text{ for all } x \leq -T$$

and

$$|v_n(x) - 1| \leq a \text{ for all } x \geq T.$$

Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, we define

$$x_1 := \sup\{x \in \mathbb{R} \text{ such that } |u_n(x) + 1| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\},$$

and

$$x_2 := \inf\{x \in \mathbb{R} \text{ such that } |u_n(x) - 1| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\}.$$

Observe that as $u_n \in \mathcal{H}$ and $\mathcal{F}_g(u_n) < +\infty$, x_1 and x_2 are real numbers. Basically, this part of the proof consists in showing that the length $x_2 - x_1$ can be controlled uniformly in $n \in \mathbb{N}$. Then choosing $\varepsilon > 0$ sufficiently small, pieces of orbits close to ± 1 on $]-\infty, x_1[$ and $]x_2, +\infty[$ can be glued to $u_{n|[x_1,x_2]}$ without increasing the action above $\mathcal{F}_g(u_n)$. This rather simple argument all the same requires some technical adjustments. We first focus on the estimate of the length $x_2 - x_1$.

Step 1 - A bound for $x_2 - x_1$. For each $0 < \varepsilon < 1$, there exists $T_\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $x_2 - x_1 \leq 2T_\varepsilon$.

Let us define $N_\varepsilon^\pm :=] \pm 1 - \varepsilon, \pm 1 + \varepsilon[$, $N_\varepsilon := N_\varepsilon^- \cup N_\varepsilon^+$ and consider the set

$$Z := \{x \in [x_1, x_2] \mid u_n(x) \in \ N_\varepsilon\}.$$

Observe that Z is a union of intervals I_i on which $|u_n'| \geq \varepsilon$. In the sequel, we assume that these intervals I_i are of maximal length. As $|u_n'(x)| \geq \varepsilon$ on any interval I_i, we infer that

$$|I_i|\varepsilon \leq \left| \int_{I_i} u_n'(x)\, dx \right| \leq 2\varepsilon$$

so that $|I_i| \leq 2$. Further except maybe for the last one, each interval I_i is followed by an interval $J_i = [c_i, d_i]$ that we also suppose to be of maximal length and which is so that one of the following conditions holds for all $x \in [c_i, d_i]$:

$$\begin{array}{llll} \text{(a)} & u_n(x) \geq 1 + \varepsilon, & u_n'(c_i) \ \geq \varepsilon, & u_n'(d_i) \ \leq -\varepsilon, \\ \text{(b)} & -1 + \varepsilon \leq u_n(x) \leq 1 - \varepsilon, & |u_n'(c_i)| \geq \varepsilon, & |u_n'(d_i)| \geq \varepsilon, \\ \text{(c)} & u_n(x) \leq -1 - \varepsilon, & u_n'(c_i) \ \leq -\varepsilon, & u_n'(d_i) \ \geq \varepsilon. \end{array}$$

Consider an interval $J_i = [c_i, d_i]$ such that (a) or (c) hold. We then obtain the estimate

$$2\varepsilon \leq |u_n'(d_i) - u_n'(c_i)| = \left| \int_{c_i}^{d_i} u_n''(x)\, dx \right| \leq \|u_n''\|_{L^2(c_i,d_i)} \sqrt{d_i - c_i}$$

and we thus infer that

$$\int_{c_i}^{d_i} \left(\frac{1}{2}(u_n'')^2 + f(u_n) \right) dx \geq \frac{2\varepsilon^2}{d_i - c_i} + r_\varepsilon(d_i - c_i) \geq 2\varepsilon\sqrt{2r_\varepsilon}, \quad (2.15)$$

where $r_\varepsilon := \inf\{f(u) \mid u \notin N_\varepsilon\}$. Notice that for an interval J_i of type (b) with $u_n'(c_i)u_n'(d_i) < 0$, the same inequality holds. At last, assume that J_i is of type (b) and such that $u_n'(c_i)u_n'(d_i) > 0$. Then we obtain

$$2 - 2\varepsilon = \left| \int_{c_i}^{d_i} u_n'(x)\, dx \right| \leq L(d_i - c_i)$$

and

$$\int_{c_i}^{d_i} \left(\frac{1}{2}(u_n'')^2 + f(u_n) \right) dx \geq r_\varepsilon(d_i - c_i) \geq \frac{r_\varepsilon(2-2\varepsilon)}{L}, \tag{2.16}$$

where L is the uniform bound on $\|u_n'\|_\infty$ obtained in Part 1. Let us denote by k the number of intervals J_i. Taking the estimates (2.15) and (2.16) into account, we infer that

$$\begin{aligned} C \geq \mathcal{F}_g(u_n) &\geq \sum_{i=1}^{k} \int_{c_i}^{d_i} \left(\frac{1}{2}(u_n'')^2 + f(u_n) \right) dx \\ &\geq k \, \min(2\varepsilon\sqrt{2r_\varepsilon}, \frac{r_\varepsilon(2-2\varepsilon)}{L}) \end{aligned} \tag{2.17}$$

so that k is uniformly bounded with respect to $n \in \mathbb{N}$.

We are now able to conclude our first step. Indeed, setting

$$\tilde{Z} = [x_1, x_2] \setminus Z,$$

we have

$$C \geq \mathcal{F}_g(u_n) \geq \int_{\tilde{Z}} f(u_n(x))\, dx \geq r_\varepsilon |\tilde{Z}|$$

and since Z is the union of at most $k+1$ intervals of length smaller than 2, we finally deduce that

$$x_2 - x_1 = |Z| + |\tilde{Z}| \leq 2(k+1) + \frac{C}{r_\varepsilon} =: 2T_\varepsilon.$$

Step 2 - Modification of u_n in $]-\infty, x_1]$ and $[x_2, +\infty[$.

We focus on the modification in $]-\infty, x_1]$. Consider the functional

$$\mathcal{F}_g|_{-\infty}^{x_1}(u) = \int_{-\infty}^{x_1} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx$$

having as domain

$$\mathcal{H}^n_{]-\infty, x_1]} := \{u \in \mathcal{H} \mid u = u_n \text{ on } [x_1, +\infty[\}.$$

We also introduce the set

$$\mathcal{D}^n_{]-\infty, x_1]} := \{u \in \mathcal{H}^n_{]-\infty, x_1]} \mid \text{ for all } x \leq x_1,\ u(x) \in [-1-a, -1+a]\}.$$

Let us fix again the notation $N_a^- :=]-1-a, -1+a[$. We then define

$$\gamma := \sup\{g(u) \mid u \in N_a^-\} \quad \text{and} \quad \eta := \inf\{f(u) \mid u \in N_a^- \setminus N_{\frac{a}{2}}^-\} > 0.$$

We first derive a lower estimate on the action of a function $u \in \mathcal{H}^n_{]-\infty,x_1]}$ whose graph does not stay in the strip $]-\infty, x_1] \times N_a^-$.

Claim 1. If $u \in \mathcal{H}^n_{]-\infty,x_1]} \setminus \mathcal{D}^n_{]-\infty,x_1]}$ *and* $\|u'\|_\infty \leq L$ *then*

$$\mathcal{F}_g|_{-\infty}^{x_1}(u) \geq \frac{\eta a}{2L}.$$

As there exists $x \leq x_1$ such that $u(x) \notin N_a^-$, we may either find $s_1 \leq s_2 \leq x_1$ such that

$$u(s_1) = -1 + \frac{a}{2}, \quad u(s_2) = -1 + a$$

and for all $x \in [s_1, s_2]$,

$$u(x) \in [-1 + \frac{a}{2}, -1 + a]$$

or $s_3 \leq s_4 \leq x_1$ verifying

$$u(s_3) = -1 - a, \quad u(s_4) = -1 - \frac{a}{2}$$

and for all $x \in [s_3, s_4]$,

$$u(x) \in [-1 - a, -1 - \frac{a}{2}].$$

Let us for instance consider the first possibility, the second being similar. We then have

$$L(s_2 - s_1) \geq \int_{s_1}^{s_2} u'(x)\, dx = u(s_2) - u(s_1) = \frac{a}{2}$$

and

$$\int_{-\infty}^{x_1} \left(\frac{u''^2}{2} + f(u) \right) dx \geq \int_{s_1}^{s_2} f(u)\, dx \geq \frac{\eta a}{2L}.$$

so that the claim follows.

Claim 2. There exists $R > 0$ *such that for all* $\varepsilon > 0$,

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_g|_{-\infty}^{x_1} \leq R\varepsilon^2.$$

For a function $u \in \mathcal{D}^n_{]-\infty,x_1]}$, we have by virtue of (B2)

$$\mathcal{F}_g|_{-\infty}^{x_1}(u) \leq \int_{-\infty}^{x_1} \left(\frac{1}{2}(u''^2 + \gamma u'^2) + \alpha(u+1)^2 \right) dx.$$

Let us define the functional

$$\mathcal{F}_{\gamma,\alpha}|_{-\infty}^{x_1}(u) := \int_{-\infty}^{x_1} \left(\frac{1}{2}(u''^2 + \gamma u'^2) + \alpha(u+1)^2 \right) dx$$

on $\mathcal{H}^n_{]-\infty,x_1]}$. Let P be the third degree polynomial satisfying

$P(x_1 - 1) = -1,\ P'(x_1 - 1) = 0,\ P(x_1) = u_n(x_1)$ and $P'(x_1) = u_n'(x_1)$.

The function $v :]-\infty, x_1] \to \mathbb{R}$ defined by

$$v(x) := \begin{cases} 0 & \text{if } x < x_1 - 1 \\ P(x) & \text{if } x_1 - 1 \le x \le x_1 \end{cases}$$

actually belongs to $\mathcal{D}^n_{]-\infty,x_1]}$ if ε is taken sufficiently small. A straightforward computation then shows that there is a constant $C(\gamma,\alpha)$ such that

$$\mathcal{F}_{\gamma,\alpha}|_{-\infty}^{x_1}(v) \le C(\gamma,\alpha)\left((u_n(x_1)+1)^2 + u_n'(x_1)^2\right) \le 2C(\gamma,\alpha)\varepsilon^2$$

so that we obtain the estimate

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_g|_{-\infty}^{x_1} \le \inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_{\gamma,\alpha}|_{-\infty}^{x_1} \le 2C(\gamma,\alpha)\varepsilon^2.$$

Conclusion of Step 2 - Let us choose $\varepsilon > 0$ sufficiently small in order to have

$$\varepsilon^2 \le \frac{\eta a}{4C(\gamma,\alpha)L}.$$

If $u_n \notin \mathcal{D}^n_{]-\infty,x_1]}$, we infer from the above estimates that we can replace u_n by $v_n \in \mathcal{D}^n_{]-\infty,x_1]}$ such that $\mathcal{F}_g(v_n) \le \mathcal{F}_g(u_n)$.

If $|u_n(x) - 1| \not\le a$ for $x \ge x_2$ we proceed in the same way to modify u_n for $x \ge x_2$.

We are now in a position to complete the proof of Part 2. Indeed, we deduce from Step 1 and Step 2 that there exist $T > 0$ and a minimizing sequence $(v_n)_n$ such that for all $n \in \mathbb{N}$, there exist $x_1 < x_2$ satisfying $x_2 - x_1 \le 2T$ and

$$|v_n(x) + 1| \le a \text{ for all } x \le x_1,$$

$$|v_n(x) - 1| \le a \text{ for all } x \ge x_2.$$

Translating v_n if necessary, we can assume $[x_1, x_2] \subset [-T, T]$ so that the conclusion of the second part follows. Observe that as $\mathcal{F}_g(v_n) \le \mathcal{F}_g(u_n)$, the sequence $(v_n)_n$ satisfies the a priori bound derived in the first part. This ends the proof. □

2.3. Qualitative Properties of the Minimizers

In this section, we study the qualitative properties of the minimizers. We have seen that when considering the model functional $\mathcal{F}_\beta$, any minimizer is odd and positive in $]0, +\infty[$. This is also true for any minimizer of $\mathcal{F}_g$ if f and g are even functions. Without this symmetry assumption, none of these characteristics are true in general. However, a generic property holds true for minimizers of $\mathcal{F}_g$ without any specific assumption. Namely, the transitions from one equilibrium to the other are monotone, which means none of the local minima or maxima is achieved in the region $[-1,1]$ between two visit of an equilibrium, except maybe before the first visit of $+1$ and after the last visit of -1.

2.3.1. Clipping. We recall the *clipping* procedure as introduced in W. D. Kalies et al. [**54**]. To prove that a minimizer only performs monotone transitions, we show that any oscillation in the range $[-1,1]$ can be discarded in such a way that the resulting function has a lower action. When dealing with a second order equation and its associated functional, we may simply cut the graph into monotone pieces, throw away unnecessary cuts and glue together the remaining parts to build a continuous monotone function. As our functional $\mathcal{F}_g$ requires square integrable second order derivative, the problem is more delicate. For example, any modification has to keep the functions at least C^1. Let us describe admissible cutoffs.

DEFINITIONS 2.10. *Let $u \in C^1[a,b]$. If u is such that $u(\alpha) = u(\beta)$ and $u'(\alpha) = u'(\beta)$ for some $\alpha < \beta$ in $[a,b]$, we say that the interval $[\alpha,\beta]$ can be clipped out, meaning that we can define a C^1 function $\hat{u}$ on the interval $[a, b-(\beta-\alpha)]$ which coincides with u and the $\beta-\alpha$ translate of $u_{|[\beta,b]}$ respectively on the intervals $[a,\alpha]$ and $[\alpha, b-(\beta-\alpha)]$. The function $\hat{u}$ is defined by*

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in [a,\alpha], \\ u(x+\beta-\alpha) & \text{if } x \in [\alpha, b-(\beta-\alpha)]. \end{cases}$$

We say that $\hat{u}$ is a clip of u.

The clipping process is depicted in Figure 2.1 below. Observe that in case the Lagrangian

$$L_g(u,u',u'') = \frac{1}{2}(u''^2 + g(u)u'^2) + f(u)$$

is non-negative, the clipping process has the nice property that if $u \in \mathcal{H}$ and $\hat{u} \in \mathcal{H}$ is a clip of u, then

$$\mathcal{F}_g(\hat{u}) \leq \mathcal{F}_g(u).$$

The following lemma gives the basic tool for clipping functions. The arguments are taken from W. D. Kalies et al. [**54**].

LEMMA 2.11. *Let $a \leq s_1 < s_2 < s_3 < s_4 \leq b$ and let $u \in H^2(a,b)$ be invertible on $[s_1,s_2]$ and $[s_3,s_4]$ and such that*

$$u(s_1) = u(s_3),\ u(s_2) = u(s_4),\ (u'(s_1) - u'(s_3))(u'(s_2) - u'(s_4)) \leq 0.$$

Then there exist $\alpha \in [s_1,s_2]$, $\beta \in [s_3,s_4]$ such that the interval $[\alpha,\beta]$ can be clipped out.

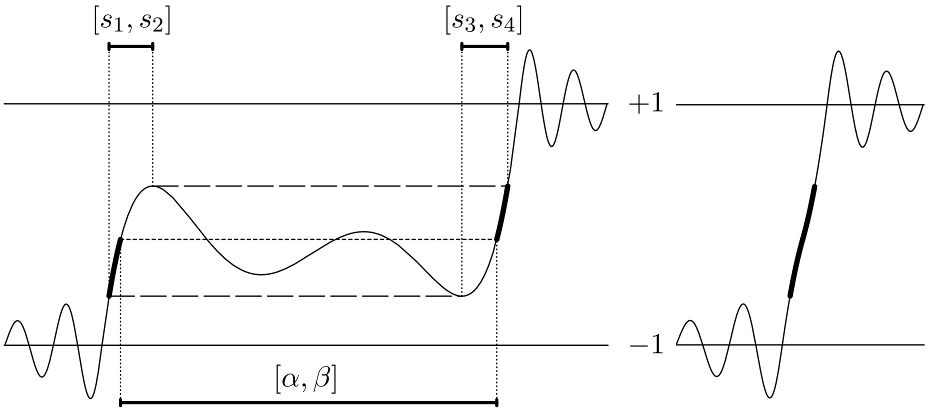


FIGURE 2.1. The clipping process. *Lemma* 2.11 *applies with* $s_1 < s_2 < s_3 < s_4$ *so that a subinterval* $[\alpha, \beta]$ *can be clipped out.*

PROOF. Let us denote by v and w the inverse of u on $[s_1, s_2]$ and $[s_3, s_4]$ respectively. By assumption, v and w are continuous so that the function $\varphi : [u(s_1), u(s_2)] \to \mathbb{R}$ defined by

$$\varphi(x) = u'(v(x)) - u'(w(x))$$

is continuous. As

$$\varphi(u(s_1)) = u'(s_1) - u'(s_3) \quad \text{and} \quad \varphi(u(s_2)) = u'(s_2) - u'(s_4),$$

we observe that

$$\varphi(u(s_1))\varphi(u(s_2)) \leq 0.$$

The continuity of φ then implies the existence of $u_0 \in [u(s_1), u(s_2)]$ such that $\varphi(u_0) = 0$. Now, taking $\alpha = v(u_0)$ and $\beta = w(u_0)$, we deduce that $s_1 \leq \alpha \leq s_2$, $s_3 \leq \beta \leq s_4$, $u(\alpha) = u(\beta) = u_0$ and $u'(\alpha) = u'(\beta)$. The interval $[\alpha, \beta]$ can therefore be clipped out. □

A typical example where Lemma 2.12 applies is displayed in Figure 2.1. Though the preceding Lemma is sufficient in practice for our purposes, we next present a slight generalization. In this variant we can drop the invertibility assumption on u in the intervals $[s_1, s_2]$ and $[s_3, s_4]$.

LEMMA 2.12. *Let* $s_1 < s_2 < s_3 < s_4$ *and let* $u \in C^1([s_1, s_4])$ *be such that*

$$u(s_1) = u(s_3), \; u(s_2) = u(s_4), \; (u'(s_1) - u'(s_3))(u'(s_2) - u'(s_4)) \leq 0.$$

Assume moreover that

$$u(s_1) < u(s) < u(s_2) \text{ for all } s \in]s_1, s_2[,$$

and

$$u(s_3) < u(s) < u(s_4) \text{ for all } s \in]s_3, s_4[.$$

Then there exist $\alpha \in [s_1, s_2]$, $\beta \in [s_3, s_4]$ *such that the interval* $[\alpha, \beta]$ *can be clipped out.*

PROOF. Consider the set

$$E := \{(x, y) \in [s_1, s_2] \times [s_3, s_4] \mid u(x) = u(y)\}.$$

It follows from degree arguments that there exists a connected set $H \subset E$ that contains (s_1, s_3) and (s_2, s_4). Next we define a function $\varphi : H \to \mathbb{R}$ by

$$\varphi(x, y) = u'(x) - u'(y).$$

Since

$$\varphi(s_1, s_3)\varphi(s_2, s_4) \leq 0,$$

the continuity of φ leads to the conclusion. □

2.3.2. Monotonicity of the Transitions. Consider the functional $\mathcal{F}_g$ defined by (2.8), assuming that g is a non-negative function and f satisfies assumptions (B1), (B2) and (B3). Then any minimizer of $\mathcal{F}_g$ in $\mathcal{H}$ (defined by (2.11)) is monotone in its transitions. If $u \in \mathcal{H}$, we define

$$x_{+1} := \inf\{x \in \mathbb{R} \mid u(x) = +1\}$$

and (2.18)

$$x_{-1} := \sup\{x \in \mathbb{R} \mid u(x) = -1\}.$$

Notice that it may happen that $x_{+1} = +\infty$ and $x_{-1} = -\infty$.

PROPOSITION 2.13. *Assume* $g \in C^2(\mathbb{R})$ *is a non-negative function and* $f \in C^2(\mathbb{R})$ *satisfies hypothesis* (B1). *Let* u *be a minimizer of the functional* $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *defined by* (2.8) *and* (2.11). *If* $x_{+1} = +\infty$ *and* $x_{-1} = -\infty$, *then* u *is strictly monotone increasing. If* $x_{+1} = +\infty$ *and* $x_{-1} > -\infty$, *respectively* $x_{+1} < +\infty$ *and* $x_{-1} = -\infty$, *then* u *is strictly monotone increasing after* x_{-1}, *respectively before* x_{+1}. *If both values are finite, then* u *has a discrete set of critical points* $\{\xi_1, \ldots, \xi_{2n} \mid n \in \mathbb{N}\}$ *in the interval* $[x_{+1}, x_{-1}]$. *Moreover, these are all local extrema such that* $u(\xi_{2i-1}) > 1$ *and* $u(\xi_{2i}) < -1$, *for* $i = 1, \ldots, n$.

PROOF. CASE 1. Let us first focus on the situation where $x_{+1} = +\infty$ and $x_{-1} = -\infty$. Assume u achieves a critical value at some point $x_0 \in \mathbb{R}$.

Claim 1. We claim $u''(x_0) \neq 0$. From Proposition 2.7, we infer that u is in the zero energy manifold, i.e.

$$u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u) = 0.$$

Since f vanishes only at ± 1 and $|u(x_0)| < 1$, it follows that $u''(x_0) \neq 0$. In particular, x_0 is an isolated local extremum.

We may now assume without loss of generality that $u(x_0)$ is a maximum. Define $s_2 := x_0$,

$$s_4 := \max\{x \geq x_0 \mid u(x) = u(s_2)\},$$
$$s_3 := \max\{x \in [s_2, s_4] \mid u'(x) = 0\},$$

and take

$$s_1 := \max\{x \leq s_2 \mid u(x) = u(s_3)\}.$$

As Lemma 2.12 applies to the function u restricted to the interval $[s_1, s_4]$, we may discard the restriction of u to some interval $[\alpha, \beta]$ containing $[s_2, s_3]$. As the discarded piece of graph has a positive contribution to the functional, we deduce that $\hat{u}$, the clip of u, has a strictly lower action than u which is a contradiction.

Case 2. When one of the values x_{+1} or x_{-1} is finite and the other is either $+\infty$ or $-\infty$, we may argue as in Case 1.

Case 3. Assume now that we are in the case where $x_{+1} < +\infty$ and $x_{-1} > -\infty$. We first prove that the critical points are local extrema.

Claim 1. If $u'(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then $u''(x_0) \neq 0$. Arguing as in Case 1, we deduce that the claim holds if $u(x_0) \neq \pm 1$. Assume $u(x_0) = +1$. Then, we may extend u by $+1$ at the right of x_0, namely we define $\bar{u} \in \mathcal{H}$ by

$$\bar{u}(x) := \begin{cases} u(x) & \text{if } x \leq x_0, \\ +1 & \text{if } x_0 < x. \end{cases}$$

If $\bar{u} = u$ then we obtain a contradiction as the equilibrium $(+1, 0, 0, 0)$ cannot be reached in a finite time. Otherwise, $\bar{u}$ has a lower action than u, which is also impossible. Clearly, if $u(x_0) = -1$, we come out with a similar contradiction.

In particular, Claim 1 implies that the critical points are isolated ($\pm\infty$ are the only possible accumulation points) and in finite number in $[x_{+1}, x_{-1}]$. We denote the set of critical points in $[x_{+1}, x_{-1}]$ by

$$\Xi := \{\xi_1, \ldots, \xi_m, \, m \in \mathbb{N}\}.$$

Claim 2. Let $\ell \in \mathbb{R}$. If $u(x) \geq \ell$ or $u(x) \leq \ell$ in the interval $[x_1, x_2]$ and $u(x_1) = u(x_2) = \ell$, then u achieves a unique local extremum in the interval $]x_1, x_2[$. Let us assume $u(x) \geq \ell$ in $[x_1, x_2]$. The other case is similar. Let $c \in]x_1, x_2[$ be such that

$$u(c) = \max_{x \in [x_1, x_2]} u(x).$$

Then u is increasing in $[x_1, c]$ and decreasing in $[c, x_2]$. Otherwise, we can either find

$$s_3 := \max\{\xi \in \Xi \cap]x_1, c[\,\} \quad \text{or} \quad \tilde{s}_2 := \min\{\xi \in \Xi \cap]c, x_2[\,\}.$$

Suppose s_3 exists. Then we define

$$s_1 := \min\{x \leq s_3 \mid u(x) = u(s_3)\},$$
$$s_2 := \min\{x \in [s_1, s_3] \mid u'(x) = 0\},$$

and take $s_4 \in [s_3, c]$ such that $\mathrm{u}(s_4) = u(s_2)$. By means of Lemma 2.12, we may now clip an interval $[\alpha, \beta]$ containing $[s_2, s_3]$. As the clip of u has a lower action, we obtain a contradiction. In the case where only $\tilde{s}_2$ can be found, we define

$$\tilde{s}_4 := \max\{x \in [\tilde{s}_2, x_2] \mid u(x) = u(\tilde{s}_2)\},$$
$$\tilde{s}_3 := \max\{x \in [\tilde{s}_2, \tilde{s}_4] \mid u'(x) = 0\},$$

and take $\tilde{s}_1 \in [c, \tilde{s}_2]$ such that $\mathrm{u}(\tilde{s}_1) = u(\tilde{s}_3)$. Applying the clipping process in the interval $[\tilde{s}_1, \tilde{s}_4]$, we reach the same contradiction.

Conclusion of Case 3. It follows from the arguments of Claim 1 that ± 1 cannot be critical values of u. Hence, the first extremal value $u(\xi_1)$ is a local maximum above $+1$ and the last one $u(\xi_m)$ is a local minimum below -1.

We claim that for all $i = 1, \ldots, m$, we have $|u(\xi_i)| > 1$. Assume that $|u(\xi_i)| \leq 1$ for some $i = 2, \ldots, m-1$. We may suppose without loss of generality that $\ell := u(\xi_i)$ is a local minimum, the other case being treated in a similar way. Since $\xi_i \in]x_{+1}, x_{-1}[$ and $\ell \in [-1, 1]$, we are able to find

$$x_1 := \max\{x < \xi_{i-1} \mid u(x) = \ell\} \text{ and } x_2 := \min\{x > \xi_{i+1} \mid u(x) = \ell\}.$$

It follows that $u(x) \geq \ell$ in $[x_1, x_2]$ and $u(x_1) = u(x_2) = \ell$. Since u has at least three critical points in $]x_1, x_2[$, we obtain a contradiction with Claim 2. This completes the proof of Case 3. □

Observe that Proposition 2.13 remains valid if f and g are of class C^1 with local Lipschitz first derivative.

2.3.3. Oscillations in the Tails. Consider a minimizer of $\mathcal{F}_g$ in $\mathcal{H}$. When either x_{-1} or x_{+1} is finite, something else can be said about the shape of the tails. In the interval $[x_{-1}, +\infty[$, the successive local maxima of the minimizer decrease to $+1$, while the successive local minima increase to $+1$. A similar statement holds in the interval $]-\infty, x_{+1}]$.

To obtain more information about the profile of the tails, we have to use non-variational arguments. As pointed out in the introduction, when ± 1 are saddle-nodes or saddle-foci, the linear flows (corresponding to the linearizations around the equilibria) and the nonlinear flow are conjugate in neighbourhoods of the equilibria. This generic result suggest that some information about the behaviour, close to the equilibria, of the solutions of the nonlinear equation can be obtained through an analysis of the linear flow. In the saddle-foci case, this lead to the following observation. If $+1$ is a saddle-focus stationary point, then the solutions of (2.9) that converge to $+1$ in the phase-space, do oscillate around the equilibrium in their tails. Obviously, the same corresponding property holds true if -1 is a saddle-focus. To simplify the notation, we assume the equilibrium has been translated to 0. Consider the linearization of (2.9)

$$z'''' - g(0)z'' + f''(0)z = 0. \tag{2.19}$$

Lemma 2.14. *Let f, $g \in C^2(\mathbb{R})$ and $f(0) = f'(0) = 0$. Assume in addition that $g(0)^2 < 4f''(0)$, i.e. $u = 0$ is a saddle-focus equilibrium of equation* (2.19)*. Then, there exists $\Delta > 0$ such that for any non-trivial solution $\hat{u}$ of* (2.9) *that satisfies*

$$\lim_{x\to+\infty} (\hat{u}(x), \hat{u}'(x), \hat{u}''(x), \hat{u}'''(x)) = (0, 0, 0, 0), \tag{2.20}$$

there exists $R \in \mathbb{R}^+$ such that for all $x_0 \geq R$, $\hat{u}$ changes sign in the interval $[x_0, x_0 + \Delta]$.

Proof. Let $\hat{u}$ be a solution of (2.9) that satisfies (2.20). Notice that the characteristic values of the linear equation (2.19) read $\pm\rho \pm i\omega$. We then choose $\Delta = 2\pi/\omega$. Let $x_0 \in \mathbb{R}$ and $z_0 \in \mathbb{R}^4$. By the choice of Δ, the solution z of (2.19) with initial condition

$$(z(x_0), z'(x_0), z''(x_0), z'''(x_0)) = z_0$$

satisfies

$$\max_{x\in[x_0,x_0+\Delta]} z(x) \geq c|z_0|, \qquad \min_{x\in[x_0,x_0+\Delta]} z(x) \leq -c|z_0|$$

and

$$\|z\|_{C^2([x_0,x_0+\Delta])} \leq M|z_0|$$

for some $c > 0$ and $M > 0$ depending only on Δ, $g(0)$ and $f''(0)$. On the other hand, we can also find $N > 0$ such that the solutions of

$$w'''' - g(0)w'' + f''(0)w = h(x),$$
$$(w(x_0), w'(x_0), w''(x_0), w'''(x_0)) = 0,$$

satisfy

$$\|w\|_{C^2([x_0,x_0+\Delta])} \leq N\|h\|_{L^\infty(x_0,x_0+\Delta)}.$$

Next, we take $\delta > 0$ such that $c - \dfrac{MN\delta}{1 - N\delta} > 0$.

To fix the ideas, we denote by $u(x; x_0, u_0)$ the solution of (2.9) with initial conditions $x_0 \in \mathbb{R}$ and

$$\begin{cases} u(x_0; x_0, u_0) = u_{01}, \\ u'(x_0; x_0, u_0) = u_{02}, \\ u''(x_0; x_0, u_0) = u_{03}, \\ u'''(x_0; x_0, u_0) = u_{04}, \end{cases}$$

where $u_0 = (u_{01}, u_{02}, u_{03}, u_{04}) \in \mathbb{R}^4$. We then set

$$\tilde{u}(x;\lambda) = u(x;x_0,\lambda\hat{u}_0),$$

with $\hat{u}_0 = (\hat{u}(x_0), \hat{u}'(x_0), \hat{u}''(x_0), \hat{u}'''(x_0))$ and we define

$$p(x) = -g(\tilde{u}(x;\lambda)),$$
$$q(x) = -g'(\tilde{u}(x;\lambda))\tilde{u}'(x;\lambda),$$

and

$$r(x) = -g'(\tilde{u}(x;\lambda))\tilde{u}''(x;\lambda) - \frac{1}{2}g''(\tilde{u}(x;\lambda))\tilde{u}'^2(x;\lambda) + f''(\tilde{u}(x;\lambda)).$$

Let us now fix x_0 large enough in such a way that $|\hat{u}_0|$ is small enough to have, for $0 \le \lambda \le 1$,

$$\sup_{x\in[x_0,x_0+\Delta]} |p(x) + g(0)| \le \delta, \qquad \sup_{x\in[x_0,x_0+\Delta]} |q(x)| \le \delta$$

and

$$\sup_{x\in[x_0,x_0+\Delta]} |r(x) - f''(0)| \le \delta.$$

Observe also that $|\hat{u}_0| \ne 0$ as a non-zero solution cannot reach the equilibrium in the phase-space in a finite time. We then write

$$\hat{u}(x) = \int_0^1 \frac{d}{d\lambda}\, u(x;x_0,\lambda\hat{u}_0)\, d\lambda.$$

The function

$$\varphi(x;x_0,\hat{u}_0,\lambda) = \frac{d}{d\lambda}\, u(x;x_0,\lambda\hat{u}_0)$$

satisfies the Cauchy problem

$$\varphi'''' + p(x)\varphi'' + q(x)\varphi' + r(x)\varphi = 0,$$
$$(\varphi(x_0), \varphi'(x_0), \varphi''(x_0), \varphi'''(x_0)) = \hat{u}_0,$$

and we can write

$$\varphi = w + z.$$

Here, w solves the equation

$$w'''' - g(0)w'' + f''(0)w = -(g(0) + p(x))\varphi''(x) - q(x)\varphi'(x)$$
$$+(f''(0) - r(x))\varphi(x),$$

together with the initial conditions

$$(w(x_0), w'(x_0), w''(x_0), w'''(x_0)) = 0$$

and z is a solution of

$$z'''' - g(0)z'' + f''(0)z = 0,$$
$$(z(x_0), z'(x_0), z''(x_0), z'''(x_0)) = \hat{u}_0.$$

We next choose $\overline{x} \in [x_0, x_0 + \Delta]$ such that

$$z(\overline{x}) \ge c|\hat{u}_0|$$

and we compute

$$\hat{u} = z + \int_0^1 w \, d\lambda.$$

Notice that we also have

$$\|w\|_{C^2([x_0, x_0+\Delta])} \le N\delta \|\varphi\|_{C^2([x_0, x_0+\Delta])}$$

and we therefore obtain the estimates

$$\begin{aligned}\|w\|_{C^2([x_0, x_0+\Delta])} &\le \frac{N\delta}{1-N\delta} \|z\|_{C^2([x_0, x_0+\Delta])} \\ &\le \frac{MN\delta}{1-N\delta} |\hat{u}_0|.\end{aligned}$$

At last, we conclude that

$$\hat{u}(\overline{x}) \ge c|\hat{u}_0| - \|w\|_\infty \ge \left(c - \frac{MN\delta}{1-N\delta}\right) |\hat{u}_0| > 0.$$

Arguing in a similar way, we may also find $\underline{x} \in [x_0, x_0 + \Delta]$ such that $\hat{u}(\underline{x}) < 0$. □

PROPOSITION 2.15. *Assume $g \in C^2(\mathbb{R})$ is a non-negative function and $f \in C^2(\mathbb{R})$ satisfies* (B1). *Suppose in addition that $g(1)^2 < 4f''(1)$. Let u minimize the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by* (2.8) *and* (2.11) *and define x_{+1}, x_{-1} as in* (2.18). *Then x_{+1} is finite. Moreover, in the interval $]x_{-1}, +\infty[$, there exist two unbounded sequences $(y_n)_n$ and $(z_n)_n$ of isolated extrema. The successive local maxima $u(y_n)$ decrease to $+1$, while the successive local minima $u(z_n)$ increase to $+1$.*

An analogous proposition may be stated if -1 is a saddle-focus stationary point.

PROOF. As $+1$ is a nondegenerate minimum of f, there exist $\delta_0 > 0$, $\eta > 0$ and $\zeta > 0$ such that

$$\zeta(u-1)^2 \le f(u) \le \eta(u-1)^2 \quad \text{and} \quad |f'(u)| \le 2\eta|u-1|, \tag{2.21}$$

for $|u - 1| \le \delta_0$.

Let u minimizes $\mathcal{F}_g$ in $\mathcal{H}$ and let T be such that $|u(x) - 1| \le \delta_0$ for every $x \ge T$. Using the inequalities in (2.21) and arguing as in the proof of Proposition 2.2, it is easy to check that

$$u - 1 \in H^4(T, +\infty).$$

Hence we have

$$\lim_{x \to +\infty} (u, u', u'', u''')(x) = (+1, 0, 0, 0).$$

Applying the analysis of Lemma 2.14 to equation (2.9) around $+1$, we conclude that there exists an unbounded sequence $(x_n)_n \subset [T, +\infty[$ of transverse crossings of u with $+1$. In particular, x_{+1} is finite and there exist a sequence $(u(y_n))_n \subset [T, +\infty[$ of isolated maxima and another one

$(u(z_n))_n \subset [T, +\infty[$ of isolated minima. The fact that these extrema are isolated follows from the law of conservation of the Hamiltonian.

We claim that the successive local maxima $u(y_n)$ decrease to $+1$, while the successive local minima $u(z_n)$ increase to $+1$. Let us prove the first statement, the second being similar. Suppose for the sake of contradiction that there exist $m < n$ such that $u(y_m) \leq u(y_n)$. If equality holds, then we obviously have a contradiction as the interval $[y_m, y_n]$ may be clipped out and the clip $\hat{u}$ has a lower action than u. Suppose the inequality is strict. Now let us define $s_2 := y_m$,

$$s_4 := \max\{x \in [y_m, y_n] \mid u(x) = u(s_2)\},$$
$$s_3 := \max\{x \in [s_2, s_4] \mid u'(x) = 0\}$$

and

$$s_1 := \max\{x \leq s_2 \mid u(x) = u(s_3)\}.$$

Lemma 2.12 applies in the interval $[s_1, s_4]$ so that we come out with the by now familiar contradiction. □

2.3.4. Symmetric Functionals in the Saddle-foci Case. Combining the results of Proposition 2.13 and Proposition 2.15, we obtain a quite complete description of the profile of the minimizers when the functional is symmetric and the equilibria are saddle-foci. Namely, we are able to prove the following Corollary.

COROLLARY 2.16. *Assume f, $g \in C^2(\mathbb{R})$ are even non-negative functions and f satisfies* (B1). *Suppose in addition that $g(\pm 1)^2 < 4f''(\pm 1)$. If u minimizes the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by* (2.8) *and* (2.11)*, then u is odd and positive in $]0, +\infty[$. Moreover, both x_{-1} and x_{+1}, defined by* (2.18)*, are finite, u is monotone increasing in the interval $]x_{-1}, x_{+1}[$ and there exist two unbounded sequences $(y_n)_n \subset]x_{+1}, +\infty[$ and $(z_n)_n \subset]x_{+1}, +\infty[$ of isolated extrema. The successive local maxima $u(y_n)$ decrease to $+1$, while the successive local minima $u(z_n)$ increase to $+1$.*

□

The profile of such a minimizer is displayed in Figure 6 in the introduction.

2.4. Notes and Comments

NOTE 2.1. As already mentioned, in case $\beta \in [\sqrt{8}, +\infty[$, the heteroclinic solution of (2.1) connecting -1 to $+1$ is unique (up to translation and symmetry). The first result in this direction is due to L. A. Peletier and W. C. Troy [**78**]. They prove that the heteroclinic obtained via the shooting method is unique in the class of monotone antisymmetric functions and they conjecture that it is actually unique in the class of all functions. J. Kwapisz [**58**] and J. B. van den Berg [**114**] have then confirmed the conjecture. The proof of J. Kwapisz relies on the use of a twist-map, while the arguments of van den Berg are based on the analysis of the phase-space and more precisely of its projection into the configuration plane (u, u'). The analysis of van den Berg applies to the model equation

$$u'''' - \beta u'' + f'(u) = 0, \tag{2.22}$$

where f is a double-well potential of class C^2. Considering the set of bounded functions

$$\mathcal{B}(a,b) := \{u \in C^4(\mathbb{R}) \mid u(x) \in [a,b] \text{ for all } x \in \mathbb{R}\},$$

and the value

$$\omega(a,b) = \max\{0, \max_{u\in[a,b]} -f''(u)\},$$

his key result states that when $\beta \geq 2\sqrt{\omega(a,b)}$, bounded solutions of (2.22) in $\mathcal{B}(a,b)$ do not cross in the configuration plane (u, u'). The uniqueness of the heteroclinic solution then follows from a sharp a priori bound on the bounded solutions and an energy ordering of the bounded orbits in the configuration plane (u, u').

It is worth mentioning that the results of J. B. van den Berg[**114**] also imply that the unique heteroclinic is asymptotically stable for the corresponding evolution equation in the space of bounded uniformly continuous functions.

Note 2.2. If in addition to the hypotheses of Theorem 2.8 we suppose that f satisfies for some $0 < b < \frac{1}{2}$ and $\beta > 0$,

$$\frac{f(u)}{(u-1)^2} \geq \beta, \quad \text{for all } u \in (1-b, 1+b),$$
$$\frac{f(u)}{(u+1)^2} \geq \beta, \quad \text{for all } u \in (-1-b, -1+b),$$

then a minimizer is indeed a classical heteroclinic. For instance, in this case, any minimizer u of $\mathcal{F}_g$ in $\mathcal{H}$ satisfies $u + 1 \in L^2(\mathbb{R}^-)$ and $u - 1 \in L^2(\mathbb{R}^+)$. Arguing as in the proof of Proposition 2.2 we then conclude that $u - 1 \in H^4(\mathbb{R}^+)$ and $u + 1 \in H^4(\mathbb{R}^-)$. The limits at $\pm\infty$ for u'' and u''' follow now easily.

Notice that the above additional assumptions hold if -1 and $+1$ are nondegenerate minima of f.

Note 2.3. Not much is known about the role of the heteroclinic solutions for the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} - \beta \dfrac{\partial^2 u}{\partial x^2} + u^3 - u = 0, & 0 < x < L,\ t > 0, \\ u(t,0) = u(t,L) = 0, & t > 0, \\ \dfrac{\partial^2 u}{\partial x^2}(t,0) = \dfrac{\partial^2 u}{\partial x^2}(t,L) = 0, & t > 0, \\ u(0,x) = u_0(x), & 0 < x < L. \end{cases}$$

When $\beta \geq \sqrt{8}$, the stability of the heteroclinic has been proved by J. B. van den Berg [**114**].

Another class of solutions, namely travelling wave solutions, is important for the dynamics of the evolution equation. These solutions are waves that evolve at constant speed. Thus they can be written $u(t,x) = U(x - ct)$ for some constant c and solve the equation

$$U'''' - \beta U'' - cU' + f'(U) = 0.$$

For this last equation, the heteroclinic solutions connecting -1 to $+1$ or 0 to -1 or $+1$ are of particular interest. The problem of the existence of such solutions is not completely solved. Partial results exist for similar models with an asymmetric nonlinear term like

$$f_a'(u) = (u + a)(u^2 - 1),\ a \neq 0.$$

The case $a = 0$ can also be treated when dealing with heteroclinics connecting 0 to -1 or $+1$. For further details, we refer to [**97, 113, 115**].

Note 2.4. It was observed by L. A. Peletier et al. [**77**] that the arguments used in the proof of Theorem 2.3 can be carried over to treat sixth order bi-stable equations of the form

$$u^{(6)} + Au^{(4)} + Bu^{(2)} + u - u^3 = 0$$

provided that $A^2 < 4B$. The associated functional is then coercive and weakly lower semi-continuous.

Note 2.5. Combining the analysis of this chapter and the arguments of Section 1.2.2, we are able to obtain existence results for non-autonomous variants of equation (2.9). The extension to fourth order system is a little more tedious but would probably not carry too much difficulties. The arguments used to shorten trajectories have to be adjusted to the functional framework. Namely, pieces of graphs have to be glued more smoothly. In order to simplify explicit computation, a reasonable choice would be to work with cubic polynomials.

Chapter 3

Sign Changing Lagrangians

The success of the arguments of Chapter 2 depends on the positivity of the Lagrangian

$$L_g(u, u', u'') = \frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \tag{3.1}$$

and therefore on the sign of g. In this chapter, we examine sign changing Lagrangians. In order to deal with such Lagrangians, the main idea is to impose a condition on g that provides a lower bound on the action over all admissible functions. We then minimize the functional

$$\mathcal{F}_g(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx, \tag{3.2}$$

whose Euler-Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{3.3}$$

We consider, in Section 3.1, the following assumption, which provides a good equilibrium between the contribution of the second term of the Lagrangian and the potential energy:

(C1) there exist a function $\tilde{g} \in C(\mathbb{R})$ and some $k < 1$ such that for all $u \in \mathbb{R}$,

$$g(u) \geq \tilde{g}(u) \quad \text{and} \quad |\tilde{G}(u)| \leq k\sqrt{8f(u)},$$

where $\tilde{G}(u) := \displaystyle\int_0^u \tilde{g}(s)\, ds.$

The results of Section 3.1 were originally presented in P. Habets, L. Sanchez, M. Tarallo and S. Terracini [**49**] and generalized in D. Bonheure, L. Sanchez, M. Tarallo and S. Terracini [**18**].

As emphasized in the introduction, condition (C1) is incompatible with a function g that takes negative values everywhere. We consider negative functions g in Section 3.3. Here, we require stronger assumptions. Mainly, we only deal with symmetric Lagrangians assuming further that

(C2) for some $k > 0$, $\beta < \sqrt{8k}$ and all $u \geq 0$,
$$f(u) \geq k(u-1)^2 \quad \text{and} \quad g(u) \geq -\beta.$$

Also, we do assume that the functional is bounded from below. We prove this is the case when $g^- = \max(0, -g)$ is small. This framework has first been considered in D. Bonheure, P. Habets and L. Sanchez [**16**].

In Section 3.3, we focus on non-symmetric functionals without any restriction on the sign of g. We then assume that the minima of the potential are saddle-focus equilibria and we suppose that

(C3) there exist $\varepsilon > 0$ and a function $\tilde{g} \in C(\mathbb{R})$ bounded from below, such that $g(u) - \tilde{g}(u) \geq \varepsilon$ and
$$\inf_{\mathcal{E}} \mathcal{F}_{\tilde{g}} > -\infty,$$
where $\mathcal{E}$ is the functional space (18) defined in the introduction and $\mathcal{F}_{\tilde{g}} : \mathcal{E} \to \mathbb{R}$ is defined according to (3.2).

Assumption (C3) gives more than a lower bound on the functional. It also leads to uniform bounds on the contributions of each term of the Lagrangian. As pointed out in the introduction, up to our knowledge, this is the first time that a non-symmetric functional as $\mathcal{F}_g$ is considered with g being negative everywhere. Theorem 3.10 below improves the results of Bonheure et al. [**16**].

3.1. Functionals with Sign Changing Acceleration Coefficient

We consider a double well potential $f \in C^1(\mathbb{R})$ such that (B1), (B2) and (B3) hold. These assumptions are introduced in Chapter 2. We also assume that $g \in C^1(\mathbb{R})$ satisfies (C1).

As already underlined, assumption (B2) is automatically satisfied when ± 1 are nondegenerate minima. In this case, if f is of class C^2 in a neighborhood of ± 1 and either $g(\pm 1) < 0$ or (C1) holds with $g = \tilde{g}$, then ± 1 are saddle-foci. Indeed, as assumption (C1) implies $\tilde{G}(\pm 1) = 0$, we obtain by means of l'Hospital's rule
$$\frac{\tilde{g}^2(\pm 1)}{4f''(\pm 1)} = \lim_{u \to \pm 1} \frac{\tilde{G}^2(u)}{8f(u)} < 1,$$

so that

$$g(\pm 1)^2 \leq \tilde{g}(\pm 1)^2 < 4f''(\pm 1).$$

In order to find a weak heteroclinic solution of (3.3), we still minimize the functional $\mathcal{F}_g$ in the space

$$\mathcal{H} = \{u \in C^1(\mathbb{R}) \mid u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}),\ \lim_{x\to\pm\infty} u(x) = \pm 1\}. \quad (3.4)$$

We first observe that under assumption (C1), the action functional $\mathcal{F}_g$ is bounded from below in $\mathcal{H}$. This justifies our minimization procedure.

LEMMA 3.1. *If f, $g \in C(\mathbb{R})$ satisfy* (C1)*, then there exists a constant $s > 0$ such that for all $u \in \mathcal{H}$*

$$\mathcal{F}_g(u) \geq s \int_{\mathbb{R}} \left(\frac{u''^2}{2} + f(u) \right) dx.$$

PROOF. Let k be given by assumption (C1). Integrating by parts and using the fact that u' is bounded and $\tilde{G}(u(\pm\infty)) = \tilde{G}(\pm 1) = 0$, we obtain

$$-\int_{\mathbb{R}} \tilde{G}(u)u''\, dx = \int_{\mathbb{R}} \tilde{g}(u)u'^2\, dx.$$

For $c \in\,]k, 1[$, we then compute

$$\begin{aligned}\mathcal{F}_g(u) &\geq \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + \tilde{g}(u)u'^2) + f(u) \right) dx \\ &\geq \int_{\mathbb{R}} \left(\frac{1}{2}(1-c^2)u''^2 + \frac{1}{2}(cu'' - \frac{\tilde{G}(u)}{2c})^2 + (f(u) - \frac{\tilde{G}(u)^2}{8c^2}) \right) dx.\end{aligned}$$

Hence, by means of assumption (C1), we come out with the estimate

$$\mathcal{F}_g(u) \geq \int_{\mathbb{R}} \left(\frac{1}{2}(1-c^2)u''^2 + (1 - \frac{k^2}{c^2})f(u) \right) dx,$$

so that the conclusion follows. □

Roughly speaking, the preceding Lemma implies that the contribution of the second term of the Lagrangian is controlled by the other two. This estimate allows to extend Lemma 2.6 and Proposition 2.7 to our new setting. Namely, the first derivative of any function in $\mathcal{H}$ having a finite action, vanishes at $\pm\infty$ and we can prove the following.

PROPOSITION 3.2. *Let f, $g \in C^1(\mathbb{R})$ satisfy* (B1) *and* (C1)*. If $u \in \mathcal{H}$ minimizes the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by* (3.2) *and* (3.4)*, then u is a weak heteroclinic solution of* (2.9) *connecting -1 to $+1$ and fulfilling $u'(\pm\infty) = 0$.*

The proof of this proposition repeats by now familiar arguments. We next state the main theorem of the section.

THEOREM 3.3. *Suppose that f, $g \in C^1(\mathbb{R})$ satisfy* (B1)*,* (B2)*,* (B3) *and* (C1)*. Then, the functional $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by* (3.2) *and* (3.4)*, possess a minimizer which is a weak heteroclinic solution of* (3.3) *connecting -1 to $+1$.*

To prove this theorem, we adopt the strategy established in the proof of Theorem 2.8. With this purpose in mind, we first adapt the arguments of Proposition 2.9 and then proceed to the proof itself.

PROOF. Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence and let $C > 0$ be a uniform bound for $\mathcal{F}_g(u_n)$.

PART 1 - A priori bounds. According to Lemma 3.1, the inequality

$$\mathcal{F}_g(u_n) \geq s \int_{\mathbb{R}} \left(\frac{u_n''^2}{2} + f(u_n) \right) dx$$

holds for each u_n. The arguments used in Part 1 of the proof of Proposition 2.9 may therefore be rephrased with C/s substituting C. Hence, we conclude that there exists $L > 0$ such that

$$\sup_{n \in \mathbb{N}} \|u_n\|_{C^1} \leq L.$$

On the other hand, we obviously have

$$\sup_{n \in \mathbb{N}} \left(\|u_n''\|_{L^2}, \int_{\mathbb{R}} f(u_n)\, dx \right) < +\infty.$$

PART 2 - Localization of the minimizing sequence. As in the proof of Proposition 2.9, we avoid a loss of compactness thanks to a "relocalization" of each element u_n.

Step 1. Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, we define

$$x_1 := \sup\{x \in \mathbb{R} \text{ such that } |u_n(x) + 1| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\},$$

and

$$x_2 := \inf\{x \in \mathbb{R} \text{ such that } |u_n(x) - 1| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\}.$$

The first step of Part 2 in the proof of Proposition 2.9 consists in showing that the length $x_2 - x_1$ is uniformly bounded. The arguments used therein only suffer small modifications to reach the same conclusion: for each $0 < \varepsilon < 1$, there exists $T_\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $x_2 - x_1 \leq 2T_\varepsilon$.

In the second step, we modify u_n in $]-\infty, x_1]$ and $[x_2, +\infty[$. Keeping the notations introduced in the proof of Proposition 2.9, we are able to prove that for any function $u \in \mathcal{H}^n_{]-\infty,x_1]}$, we have

$$\mathcal{F}_g|_{-\infty}^{x_1}(u) \geq s \int_{-\infty}^{x_1} \left(\frac{u''^2}{2} + f(u) \right) dx + \frac{1}{2}\tilde{G}(u(x_1))u'(x_1), \tag{3.5}$$

where, for any interval $[a,b] \subset \mathbb{R}$, we still use the notation $\mathcal{F}_g|_a^b(u)$ to denote the contribution of u to the functional $\mathcal{F}_g$ in this interval. Indeed, this follows arguing as in Lemma 3.1 and integrating by parts.

Next, we rephrase Claim 1 of the proof of Proposition 2.9 as follows.

Claim 1. If $u \in \mathcal{H}^n_{]-\infty,x_1]} \setminus \mathcal{D}^n_{]-\infty,x_1]}$ and $\|u'\|_\infty \le L$ then

$$\mathcal{F}_g|_{-\infty}^{x_1}(u) \ge \frac{s\eta a}{2L} - \delta\varepsilon,$$

where

$$\delta := \sup\{|\tilde{G}(u)| \mid u \in N_a^-\} \ge 0.$$

The proof of this claim is identical to that in the proof of Proposition 2.9 using in addition estimate (3.5).

Claim 2 is unchanged with regards to that in the proof of Proposition 2.9.

Claim 2. There exists $R > 0$ so that for all $\varepsilon > 0$,

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_g|_{-\infty}^{x_1} \le R\varepsilon^2.$$

Step 2. Now, let us choose $\varepsilon > 0$ sufficiently small in order to have

$$\mathcal{F}_g|_{-\infty}^{x_1}(u) \ge \frac{s\eta a}{4L}$$

for $u \in \mathcal{H}^n_{]-\infty,x_1]} \setminus \mathcal{D}^n_{]-\infty,x_1]}$ satisfying $\|u'\|_\infty \le L$ and

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_g|_{-\infty}^{x_1} < \frac{s\eta a}{4L}.$$

If $u_n \notin \mathcal{D}^n_{]-\infty,x_1]}$, we infer from the above estimates that we can replace u_n by $v_n \in \mathcal{D}^n_{]-\infty,x_1]}$ such that $\mathcal{F}_g(v_n) \le \mathcal{F}_g(u_n)$.

If $|u_n(x) - 1| \not\le a$ for $x \ge x_2$ we proceed in the same way to modify u_n for $x \ge x_2$.

Step 3. From what precedes, we infer that there exist $T > 0$ and a minimizing sequence that we still denote by $(u_n)_n$ such that for all $n \in \mathbb{N}$,

$$|u_n(x) + 1| \le a \text{ for all } x \le -T$$

and

$$|u_n(x) - 1| \le a \text{ for all } x \ge T.$$

Indeed, translating u_n if necessary, we may assume $[x_1, x_2] \subset [-T, T]$.

Observe also that after the possible modifications of Step 2, the new minimizing sequence still satisfies the a priori bound derived in Part 1.

PART 3 - Existence of a minimizer. It follows from the a priori bounds that $(u_n)_n$ has a subsequence (still written $(u_n)_n$ for simplicity) such that for some function u

$$u_n \overset{C^1_{\mathrm{loc}}(\mathbb{R})}{\longrightarrow} u, \quad u''_n \overset{L^2(\mathbb{R})}{\rightharpoonup} u''.$$

Working as in the proof of Theorem 2.8, we come to the conclusion that $u \in \mathcal{H}$. To see that u is a minimizer of $\mathcal{F}_g$, arguing as in Lemma 3.1, we compute

$$\begin{aligned}
\mathcal{F}_g(u_n) &= \int_{\mathbb{R}} \left(\frac{1}{2}(u_n''^2 + \tilde{g}(u_n)u_n'^2) + f(u_n) \right) dx \\
&\quad + \int_{\mathbb{R}} \frac{1}{2} \left(g(u_n) - \tilde{g}(u_n) \right) u_n'^2 \, dx \\
&= \int_{\mathbb{R}} \frac{1}{2} \left(u_n'' - \frac{\tilde{G}(u_n)}{2} \right)^2 dx + \int_{\mathbb{R}} \left(f(u_n) - \frac{\tilde{G}(u_n)^2}{8}) \right) dx \qquad (3.6) \\
&\quad + \int_{\mathbb{R}} \frac{1}{2} \left(g(u_n) - \tilde{g}(u_n) \right) u_n'^2 \, dx.
\end{aligned}$$

As all the terms on the right-hand side are positive, we deduce that

$$\sup_{n \in \mathbb{N}} \left\| u_n'' - \frac{\tilde{G}(u_n)}{2} \right\|_{L^2} < +\infty.$$

Going to a subsequence if necessary, we may assume that $u_n'' - \tilde{G}(u_n)/2$ converges weakly in L^2 and therefore

$$\int_{\mathbb{R}} \frac{1}{2}(u'' - \frac{\tilde{G}(u)}{2})^2 \, dx \le \liminf_{n \to +\infty} \int_{\mathbb{R}} \frac{1}{2}(u_n'' - \frac{\tilde{G}(u_n)}{2})^2 \, dx.$$

Now observe that the integrands of the two last terms of (3.6) converge for each $x \in \mathbb{R}$ and are positive. Fatou's lemma is then applicable and we obtain

$$\int_{\mathbb{R}} \left(f(u) - \frac{\tilde{G}(u)^2}{8} \right) dx \le \liminf_{n \to +\infty} \int_{\mathbb{R}} \left(f(u_n) - \frac{\tilde{G}(u_n)^2}{8} \right) dx$$

and

$$\int_{\mathbb{R}} \frac{1}{2}(g(u) - \tilde{g}(u))u'^2 \, dx \le \liminf_{n \to +\infty} \int_{\mathbb{R}} \frac{1}{2}(g(u_n) - \tilde{g}(u_n))u_n'^2 \, dx.$$

As $u \in \mathcal{H}$, equality (3.6) also holds with u_n replaced by u. We finally deduce that

$$\mathcal{F}_g(u) \le \liminf_{n \to +\infty} \mathcal{F}_g(u_n) = \inf_{\mathcal{H}} \mathcal{F}_g$$

which obviously implies

$$\mathcal{F}_g(u) = \inf_{\mathcal{H}} \mathcal{F}_g.$$

The fact that u is a weak heteroclinic solution of (3.3) follows from Proposition 3.2. $\square$

We close this section by proving that, under assumption (C1), the clipping process still decreases the action. As a consequence, all the qualitative properties established in Section 2.3 extend to the minimizer obtained in Theorem 3.3.

Lemma 3.4. *Suppose that $f \in C(\mathbb{R})$ is a non-negative function and $g \in C(\mathbb{R})$ satisfies* (C1). *If $u \in H^2(a,b)$ and $\hat{u} \in H^2(a, b-(\beta-\alpha))$ is a clip of u, then*

$$\mathcal{F}_g|_a^{b-(\beta-\alpha)}(\hat{u}) \leq \mathcal{F}_g|_a^b(u).$$

Proof. We compute

$$\mathcal{F}_g|_a^{b-(\beta-\alpha)}(\hat{u}) = \mathcal{F}_g|_a^{\alpha}(u) + \mathcal{F}_g|_{\beta}^{b}(u) = \mathcal{F}_g|_a^b(u) - \mathcal{F}_g|_{\alpha}^{\beta}(u)$$

and arguing as in Lemma 3.1, we infer that

$$\mathcal{F}_g|_{\alpha}^{\beta}(u) \geq s\int_{\alpha}^{\beta}\left(\frac{u''^2}{2} + f(u)\right)dx + \frac{1}{2}(\tilde{G}(u(\beta))u'(\beta) - \tilde{G}(u(\alpha))u'(\alpha)).$$

Since $u(\alpha) = u(\beta)$, $u'(\alpha) = u'(\beta)$, we obtain the inequality

$$\mathcal{F}_g|_{\alpha}^{\beta}(u) \geq 0$$

so that the result follows. $\square$

3.2. Functionals of Swift-Hohenberg Type

The compatibility condition (C1) allows g to take negative values but it then implies that g takes positive values as well. Therefore, the stationary Swift-Hohenberg equation

$$u'''' - \beta u'' + u^3 - u = 0, \tag{3.7}$$

with $\beta < 0$, is not covered by the result of the preceding section.

When stronger assumptions are imposed on the potential f, we can deal with negative functions g in $\mathcal{F}_g$. Namely, we assume that f and $g \in C^1(\mathbb{R})$ are even functions such that $f(1) = 0$ and (C2) holds.

We then look at the minimizers of the functional $\mathcal{F}_g$ in the space

$$\mathcal{E} = \{u \mid u(0) = 0,\ u+1 \in H^2(\mathbb{R}^-),\ u-1 \in H^2(\mathbb{R}^+)\}. \tag{3.8}$$

We may easily check that the statement of Proposition 3.2 holds with the setting of this section, with the additional conclusion that any minimizer is a classical heteroclinic.

Our first observation is that under assumption (C2), for small $\beta > 0$, we have

$$\inf_{u \in \mathcal{E}} \mathcal{F}_g(u) > -\infty.$$

Lemma 3.5. *Let f and $g \in C(\mathbb{R})$ be even functions such that* (C2) *holds. Then there exists $\beta_1 > 0$ such that for $\beta \le \beta_1$, we have*

$$\inf_{u\in\mathcal{E}} \mathcal{F}_g(u) \ge 0.$$

Proof. Let us first recall the following interpolation inequality, see for example R. A. Adams [**1**]. For all $\ell > 0$, there exists $C > 0$ such that for any interval I with $|I| \ge \ell$ and all $u \in H^2(I)$,

$$\|u'\|^2_{L^2(I)} \le C \left(\|u\|^2_{L^2(I)} + \|u''\|^2_{L^2(I)} \right). \tag{3.9}$$

Let u be a function in $\mathcal{E}$ and consider an interval $[a, b]$ where u is non-negative and $u(a) = u(b) = 0$. Suppose that $b - a \ge 1$. Then using the inequality (3.9) applied to $u - 1$, we obtain

$$\begin{aligned} \mathcal{F}_g|_a^b(u) &\ge \int_a^b \left(\frac{1}{2}(u''^2 - \beta u'^2) + k(u-1)^2 \right) dx \\ &\ge \frac{1-\beta C}{2} \|u''\|^2_{L^2(a,b)} + (k - \frac{\beta C}{2}) \|u - 1\|^2_{L^2(a,b)}. \end{aligned} \tag{3.10}$$

If $b - a < 1$, we compute

$$\|u'\|_{L^2(a,b)} \le \frac{b-a}{\sqrt{2}} \; \|u''\|_{L^2(a,b)}$$

and

$$\begin{aligned} \int_a^b \left(\frac{1}{2}(u''^2 - g(u)u'^2) + f(u) \right) dx &\ge \int_a^b \left(\frac{1}{2}(u''^2 - \beta u'^2) + f(u) \right) dx \\ &\ge \frac{1}{2}(1 - \frac{\beta}{2}) \|u''\|^2_{L^2(a,b)}. \end{aligned}$$

In both cases, we deduce that

$$\int_a^b \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \ge 0$$

if $\beta \le \min(2, 1/C, 2k/C)$. A similar argument holds if $[a, b]$ is an interval where u is non-positive. At last, if u is non-negative on the interval $[T, +\infty[$, we deduce as in (3.10) that

$$\int_T^{+\infty} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \ge 0.$$

Hence, the conclusion follows. □

It should be emphasized that the statement of Lemma 3.5 is not of a perturbative nature. Indeed, an estimate of β_1 may be obtained through the estimate of the best constant in the interpolation inequality (3.9). However, such an estimate of β_1 is probably far from optimal.

Next, assuming that the functional $\mathcal{F}_g$ is bounded from below, we prove the existence of a minimizer. We deduce from the previous lemma that the result holds at least when $g \ge -\beta$ with $\beta > 0$ small.

THEOREM 3.6. *Let f, $g \in C^1(\mathbb{R})$ be even functions such that* (C2) *holds and $f(1) = 0$. Assume further that the functional $\mathcal{F}_g : \mathcal{E} \to \mathbb{R}$ defined by* (3.2) *and* (3.8) *is bounded from below, i.e.*

$$\inf_{\mathcal{E}} \mathcal{F}_g > -\infty.$$

Then $\mathcal{F}_g$ has a minimizer, which is a heteroclinic solution of equation (3.3) *connecting -1 to $+1$. Moreover, any minimizer is odd and does only vanish at zero.*

PROOF. **Step 1.** We first prove the existence of an odd minimizing sequence $(u_n)_n$ such that $u_n(x) > 0$ for all $x > 0$. Let $(v_n)_n \subset \mathcal{E}$ be a minimizing sequence. Observe that if v_n is not odd, the appropriate symmetrization of v_n restricted to $\mathbb{R}^+$ or $\mathbb{R}^-$ decreases the action. Assume now that v_n has a positive zero and define

$$x_n := \sup\{x > 0 \mid v_n(x) = 0\}.$$

We claim that

$$\mathcal{F}_g|_{-x_n}^{x_n}(v_n) = \int_{-x_n}^{x_n} \left(\frac{1}{2}(v_n''^2 + g(v_n)v_n'^{\,2}) + f(v_n)\right) dx \geq 0.$$

Indeed, suppose that $\mathcal{F}_g|_{-x_n}^{x_n}(v_n) = -C < 0$ and take $j \in \mathbb{N}$ such that $\mathcal{F}_g(v_n) - 2jC < \inf_{\mathcal{E}} \mathcal{F}_g$. Define the odd function $v_n^* \in \mathcal{E}$ by

$$v_n^*(x) := \begin{cases} v_n(x - 2ix_n), \text{ if } x \in [2ix_n, (2i+1)x_n[, \; i = 0, \cdots, j, \\ -v_n(2ix_n - x), \text{ if } x \in [(2i-1)x_n, 2ix_n[, \; i = 1, \cdots, j, \\ v_n(x - 2(j+1)x_n), \text{ if } x \in [(2j+1)x_n, +\infty[. \end{cases}$$

We then obtain a contradiction since

$$\mathcal{F}_g(v_n^*) = \mathcal{F}_g(v_n) + 2j\mathcal{F}_g|_{-x_n}^{x_n}(v_n) < \mathcal{F}_g(v_n) - 2jC < \inf_{\mathcal{E}} \mathcal{F}_g.$$

Now, let $u_n \in \mathcal{E}$ be the odd function defined by $u_n(x) = v_n(x + x_n)$ for $x \geq 0$. This function u_n vanishes only at 0 and since $\mathcal{F}_g|_{-x_n}^{x_n}(v_n) \geq 0$, the sequence $(u_n)_n$ is also a minimizing sequence.

Step 2. As there exists an odd minimizing sequence, it is sufficient to minimize the restricted functional

$$\mathcal{F}_g^+(u) := \int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u)\right) dx$$

in the space of functions

$$\mathcal{E}^+ = \{u : \mathbb{R}^+ \to \mathbb{R} \mid u(0) = 0, \; u - 1 \in H^2(\mathbb{R}^+)\}.$$

We first claim that $u_n - 1$ is uniformly bounded in $H^2(\mathbb{R}^+)$. To prove this, we compute for $u \in \mathcal{E}^+$,

$$\begin{aligned}
&\int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 - \beta u'^2) + k(u-1)^2 \right) dx \\
&= \varepsilon \|u-1\|^2_{H^2(\mathbb{R}^+)} + \frac{1-2\varepsilon}{2} \int_{\mathbb{R}^+} \left(u''^2 - \beta_\varepsilon u'^2 + \frac{1}{4}\beta_\varepsilon^2 (u-1)^2 \right) dx \\
&\quad + \left(k - \varepsilon - \frac{(\beta+2\varepsilon)^2}{8(1-2\varepsilon)} \right) \int_{\mathbb{R}^+} (u-1)^2 \, dx,
\end{aligned}$$

where $\varepsilon \in]0, 1/2[$ and $\beta_\varepsilon := \frac{\beta+2\varepsilon}{1-2\varepsilon}$. Notice that we may choose ε small enough in order to have

$$k - \varepsilon - \frac{(\beta+2\varepsilon)^2}{8(1-2\varepsilon)} \geq 0.$$

Integrating by parts the second term on the right-hand side, we obtain

$$\int_{\mathbb{R}^+} \left(u'' + \frac{\beta_\varepsilon}{2}(u-1) \right)^2 dx - \beta_\varepsilon u'(0) \geq -\beta_\varepsilon u'(0).$$

As u_n is positive, we infer that

$$\begin{aligned}
\mathcal{F}_g^+(u_n) &\geq \int_{\mathbb{R}^+} \left(\frac{1}{2}(u_n''^2 - \beta u_n'^2) + k(u_n-1)^2 \right) dx \\
&\geq \varepsilon \|u_n - 1\|^2_{H^2(\mathbb{R}^+)} - \beta_\varepsilon u_n'(0).
\end{aligned}$$

We now deduce through the use of the continuous injection of $H^2(\mathbb{R}^+)$ into $C^1(\mathbb{R}^+)$ that for some $C > 0$, we have

$$\mathcal{F}_g^+(u_n) \geq \varepsilon \|u_n - 1\|^2_{H^2(\mathbb{R}^+)} - C\|u_n - 1\|_{H^2(\mathbb{R}^+)}.$$

Hence, the claim follows from the uniform bound on $\mathcal{F}_g^+(u_n)$.

We thus infer that, at least for a subsequence (that we still denote by u_n), there exists $u \in \mathcal{E}^+$ such that

$$u_n - 1 \overset{H^2(\mathbb{R}^+)}{\rightharpoonup} u - 1 \quad \text{and} \quad u_n \overset{C^1_{\text{loc}}(\mathbb{R}^+)}{\longrightarrow} u.$$

In order to see that u is a minimizer, we write

$$\begin{aligned}
\mathcal{F}_g^+(u_n) &= \frac{1}{2}\int_{\mathbb{R}^+} \left(u_n''^2 - \beta u_n'^2 + \frac{\beta^2}{4}(u_n-1)^2 \right) dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^+} (g(u_n) + \beta)\, u_n'^2 \, dx + \int_{\mathbb{R}^+} \left(f(u_n) - \frac{\beta^2}{8}(u_n-1)^2 \right) dx \\
&= \frac{1}{2}\int_{\mathbb{R}^+} \left(u_n'' + \frac{\beta}{2}(u_n-1) \right)^2 dx - \frac{\beta}{2} u_n'(0) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^+} (g(u_n) + \beta)\, u_n'^2 \, dx + \int_{\mathbb{R}^+} \left(f(u_n) - \frac{\beta^2}{8}(u_n-1)^2 \right) dx.
\end{aligned}$$

Observe then that in the last equality, the first integral is convex and that Fatou's Lemma is applicable to the last two so that

$$\mathcal{F}_g^+(u) \le \lim_{n\to+\infty} \mathcal{F}_g^+(u_n) = \inf_{\mathcal{E}^+} \mathcal{F}_g^+.$$

Now, as $u \in \mathcal{E}^+$, we conclude that $\mathcal{F}_g^+(u) = \inf_{\mathcal{E}^+} \mathcal{F}_g^+$ and still denoting by u its odd extension to $\mathbb{R}$, we have $\mathcal{F}_g(u) = \inf_{\mathcal{E}} \mathcal{F}_g$.

The fact that u is a heteroclinic solution of (3.3) follows from by now familiar arguments.

The proof that any minimizer is odd follows the lines of the third step of the proof of Theorem 2.3. □

Arguing as in this proof, it is easily shown that, under assumption (C2), any clip $\hat{u}$ of u has a lower action than u. Indeed, if the action of u on the clipped interval is negative, then, extending this piece of function by periodicity, we can build a function with arbitrarily negative action. The result of Section 2.3 applies therefore to the minimizer obtained in Theorem 3.6. In particular, when ± 1 are saddle-foci, we have a quite precise description of the profile of the minimizer (see Corollary 2.16). In the following theorem, we observe that, when ± 1 are saddle-foci, the minimum of $\mathcal{F}_g$ is non-negative.

THEOREM 3.7. *Let f and $g \in C^2(\mathbb{R})$ be even functions such that $f(1) = 0$, $g(\pm 1)^2 < 4f''(\pm 1)$ and assume that* (C2) *holds. Then, if u is a minimizer of $\mathcal{F}_g : \mathcal{E} \to \mathbb{R}$ defined by* (3.2) *and* (3.8)*, we have $\mathcal{F}_g(u) \ge 0$.*

PROOF. Let $u \in \mathcal{E}$ be such that $\mathcal{F}_g(u) = \inf_{\mathcal{E}} \mathcal{F}_g = -C$ for some $C > 0$. As u is a minimizer, it satisfies equation (3.3) and we also know that

$$\lim_{x\to+\infty} (u(x), u'(x), u''(x), u'''(x)) = (1, 0, 0, 0).$$

Observe that Theorem 3.6 implies that u is odd so that

$$\mathcal{F}_g(u) = 2\int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx = -C.$$

Since $u = \pm 1$ are saddle-focus equilibria, we deduce, by means of the analysis of Lemma 2.14, that $u(x)$ oscillates around $+1$ as $x \to +\infty$. Hence, we can find $T > 0$ large enough so that

$$\int_0^T \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \le -C/4,$$

and $u'(T)=0$. Defining the odd function $v^* \in \mathcal{E}$ by

$$v^*(x)=\begin{cases} u(x) & \text{if } x\in[0,T),\\ u(2T-x) & \text{if } x\in[T,2T),\\ -u(x-2T) & \text{if } x\in[2T,3T],\\ -u(4T-x) & \text{if } x\in[3T,4T],\\ u(x-4T) & \text{if } x\geq 4T,\end{cases}$$

we obtain

$$\begin{aligned}\mathcal{F}_g(v^*)=8\int_0^T &\left(\frac{1}{2}(u''^2+g(u)u'^2)+f(u)\right)dx\\ &+2\int_{\mathbb{R}+}\left(\frac{1}{2}(u''^2+g(u)u'^2)+f(u)\right)dx<-2C.\end{aligned}$$

This contradicts the definition of u. Hence, we have $\inf_{\mathcal{E}}\mathcal{F}_g\geq 0$. □

We next state an obvious corollary of Theorem 3.7.

COROLLARY 3.8. *Let f and $g\in C^2(\mathbb{R})$ be even functions such that $f(1)=0$ and* (C2) *holds. Assume moreover that $g(1)^2<4f''(1)$. If there exists $u\in\mathcal{E}$ such that $\mathcal{F}_g(u)<0$, then $\inf_{\mathcal{E}}\mathcal{F}_g=-\infty$.*

PROOF. Suppose by contradiction that the conclusion of the corollary is false. Then Theorem 3.6 implies the existence of a minimizer which has a negative action. This contradicts Theorem 3.7. □

Let us consider the stationary Swift-Hohenberg equation (3.7). We know from Theorem 3.6 and Lemma 3.5 that the corresponding functional

$$\mathcal{F}_\beta(u)=\int_{\mathbb{R}}\left(\frac{1}{2}(u''^2+\beta u'^2)+\frac{1}{4}(u^2-1)^2\right)dx \tag{3.11}$$

has a minimum in $\mathcal{E}$ for β greater than some small negative constant. If β becomes too negative, the minimum no longer exists. Indeed, it is shown in V. J. Mizel et al. [**72**] that taking the function

$$w(x)=A(\beta)\sin(\omega(\beta)x),$$

where $A(\beta)=\sqrt{\frac{\beta^2+4}{3}}$ and $\omega(\beta)=\sqrt{\frac{|\beta|}{2}}$, we have

$$\int_0^{2\pi/\omega(\beta)}\left(\frac{1}{2}(w''^2+\beta w'^2)+\frac{1}{4}(w^2-1)^2\right)dx<0 \tag{3.12}$$

if $\beta<\beta^*=-\sqrt{2\sqrt{6}-4}=-0.9481\ldots$. It follows that the functional $\mathcal{F}_\beta$ is unbounded from below whenever $\beta<\beta^*$. Therefore, there exists $\beta^*\leq\beta_0<0$ such that for $\beta<\beta_0$, the functional is unbounded from below. We give in the next theorem a characterization of β_0, which shows also that the minimum of $\mathcal{F}_\beta$ exists whenever $\beta\in[\beta_0,+\infty[$.

THEOREM 3.9. *Let β_0 be defined by*

$$\beta_0 = \inf\{\beta < 0 \mid \inf_{\mathcal{E}} \mathcal{F}_\beta \geq 0\}. \tag{3.13}$$

Then $\beta_0 \in [\beta^*, 0[$ and for all $\beta \geq \beta_0$, the functional $\mathcal{F}_\beta : \mathcal{E} \to \mathbb{R}$ defined by (3.11) *and* (3.8) *has a minimum, which is an odd heteroclinic solution of* (3.7) *from -1 to $+1$, with exactly one zero. Moreover, for all $\beta < \beta_0$, $\mathcal{F}_\beta$ is unbounded from below.*

PROOF. For $u \geq 0$, we have $(u^2-1)^2/4 \geq (u-1)^2/4$ so that Theorem 3.6 covers the functional $\mathcal{F}_\beta$ for $\beta > -\sqrt{2}$. As long as $|\beta|$ is small, it follows from Lemma 3.5 and Theorem 3.6 that there exists $u \in \mathcal{E}$ such that $\mathcal{F}_\beta(u) = \inf_{\mathcal{E}} \mathcal{F}_\beta$. Also we deduce from (3.12) that $\beta_0 \geq \beta^*$.

We now claim that $\inf_{\mathcal{E}} \mathcal{F}_{\beta_0} \geq 0$. If it were not the case, the same would hold true for β near enough β_0, which contradicts the definition of β_0. Next, as $\beta_0 \geq \beta^* > -\sqrt{2}$, we infer from Theorem 3.6 that for any $\beta \geq \beta_0$, $\mathcal{F}_\beta$ has an odd minimizer having exactly one zero.

It follows also from Corollary 3.8 that if $-\sqrt{2} < \beta < \beta_0$, the functional $\mathcal{F}_\beta$ is unbounded from below. In case $\beta \leq -\sqrt{2}$ the conclusion clearly follows from (3.12) arguing as in the proof of Theorem 3.3. $\square$

Numerical computations [**112**] seems to indicate that heteroclinic solutions of (3.7) exist at least until $\beta \approx -2.32$. Theorem 3.9 shows that for $\beta < \beta_0$, these critical points are not global minima.

3.3. Non-symmetric Functionals

In this section we partially relax the assumptions of Theorem 3.6. On the one hand, we drop the symmetry assumption but on the other hand, we do assume that hypothesis (C3) holds. We also require ± 1 to be saddle-foci. At first sight, these additional assumptions could seem of a technical nature. However, they turn out to be very powerful. Assumption (C3) does not only provide a lower bound on the functional, it gives a uniform lower bound for the contribution of each term of the functional. The uniform bound on the potential energy, together with an a priori estimate for the C^1-norm, implies that a quasi-minimizer do not spend too much time far away from the minima ± 1 of f. A sharp analysis of the local minimizers close to a saddle-focus then provides a bound on the time a quasi-minimizer may spend in a neighbourhood of the equilibria before leaving this neighbourhood. These estimates provide a sufficient control on a minimizing sequence.

THEOREM 3.10. *Let $f \in C^2(\mathbb{R})$ be a non-negative double well potential satisfying assumptions* (B1) *and* (B3). *Assume in addition that ± 1 are nondegenerate minima of f and $g \in C^2(\mathbb{R})$ is such that* (C3) *holds and $g(\pm 1)^2 < 4f''(\pm 1)$. Then the functional $\mathcal{F}_g : \mathcal{E} \to \mathbb{R}$ defined by*

(3.2) *and* (3.8) *has a minimizer, which is a heteroclinic solution of* (3.3) *starting from* -1 *and ending at* $+1$.

The proof of this theorem requires some preliminary results. In the next section, we essentially focus on the behaviour of the local minimizers close to a saddle-focus equilibrium.

3.3.1. Analysis of the Local Minimizers Close to a Saddle–focus Equilibrium. We first establish a useful inequality.

LEMMA 3.11. *Let* $k > 0$ *and* $\beta \in [0, \sqrt{8k})$. *Then there exists* $\varepsilon > 0$ *such that for any* $u \in H^2(a, b)$, *we have*

$$\int_a^b \left(\frac{1}{2}(u''^2 - \beta u'^2) + ku^2\right) dx \geq \varepsilon \|u\|^2_{H^2(a,b)} - \left(\varepsilon + \frac{\beta}{2}\right) [uu']_a^b.$$

PROOF. We argue as in the proof of Theorem 3.6. Notice that for any constant α,

$$\int_a^b (u'' + \alpha u)^2 \, dx = \int_a^b \left(u''^2 - 2\alpha u'^2 + \alpha^2 u^2\right) dx + 2\alpha [u'u]_a^b. \quad (3.14)$$

We then compute

$$\int_a^b \left(\frac{1}{2}(u''^2 - \beta u'^2) + ku^2\right) dx = \varepsilon \int_a^b \left(u''^2 + u'^2 + u^2\right) dx$$
$$+ \frac{1 - 2\varepsilon}{2} \int_a^b \left(u''^2 - \frac{\beta + 2\varepsilon}{1 - 2\varepsilon} u'^2 + \frac{1}{4}\left(\frac{\beta + 2\varepsilon}{1 - 2\varepsilon}\right)^2 u^2\right) dx$$
$$+ \left(k - \varepsilon - \frac{(\beta + 2\varepsilon)^2}{8(1 - 2\varepsilon)}\right) \int_a^b u^2 \, dx.$$

Now, choosing $\varepsilon \in]0, 1/2[$ small enough, we have

$$k - \varepsilon - \frac{(\beta + 2\varepsilon)^2}{8(1 - 2\varepsilon)} \geq 0.$$

Finally, using (3.14) with $2\alpha = \frac{\beta + 2\varepsilon}{1 - 2\varepsilon}$, we get the desired estimate. $\square$

Next we recall the continuous imbedding of $H^1(a, b)$ in $C(a, b)$. The dependence on the length of the interval $]a, b[$ is important in our application.

LEMMA 3.12. *Let* $-\infty \leq a < b \leq +\infty$. *There exists a positive constant* C *such that for all* $a < b$ *and all* $u \in H^1(a, b)$,

$$\sup_{x \in (a,b)} |u(x)| \leq C\left(1 + \frac{1}{b - a}\right) \|u\|_{H^1(a,b)}. \quad (3.15)$$

Since the proof follows from standard arguments that can be found in classical textbook (see e.g. H. Brezis [**20**]), we omit it.

To fix the ideas and to simplify the notation, we assume in the following lemma that f is a potential for which 0 is a nondegenerate minimum and g is such that 0 is a saddle-focus equilibrium of the linear equation

$$u'''' - g(0)u'' + f''(0)u = 0. \tag{3.16}$$

Basically, the next lemma shows that the minimizers of the functional

$$\mathcal{F}_g|_a^b(u) = \int_a^b \left(\frac{1}{2}\left((u''^2) + g(u)u'^2\right) + f(u)\right) dx \tag{3.17}$$

in the set

$$\mathcal{E}_{[a,b]}(\eta) := \{u \in \mathcal{E}_{[a,b]} \mid \|u\|_{L^\infty(a,b)} \leq \eta\}, \tag{3.18}$$

where

$$\mathcal{E}_{[a,b]} := \{u \in H^2(a,b) \mid (u(a), u'(a)) = y_0 \text{ and } (u(b), u'(b)) = y_1\},$$

are small for the C^3-norm whenever η, y_0 and y_1 are small. The arguments are adapted from those in W. D. Kalies et al. [**54**] where it is assumed that g is a non-negative function.

Lemma 3.13. *Assume that f, $g \in C^2(\mathbb{R})$ satisfy $f(0) = f'(0) = 0$, $f''(0) > 0$ and $g(0)^2 < 4f''(0)$. Then, there exist $\eta > 0$, $\delta_0 > 0$ and $S > 1$ such that if $\max(\|y_0\|, \|y_1\|) \leq \delta_0$ and $b - a \geq 1$, the functional $\mathcal{F}_g|_a^b : \mathcal{E}_{[a,b]} \to \mathbb{R}$ has a minimizer $u \in \mathcal{E}_{[a,b]}(\eta)$ such that*

$$\|u\|_{C^3([a,b])} \leq S \max(\|y_0\|, \|y_1\|). \tag{3.19}$$

Proof. Throughout the proof, C is a positive constant that may change from line to line. From the inequality $g(0)^2 < 4f''(0)$, we infer that there exist $\eta > 0$, $\beta > 0$ and $k > 0$ satisfying $\beta < \sqrt{8k}$ such that for all $u \in [-\eta, \eta]$,

$$g(u) \geq -\beta \quad \text{and} \quad f(u) \geq ku^2.$$

Step 1 - Estimate of $\inf_{\mathcal{E}_{[a,b]}(\eta)} \mathcal{F}_g|_a^b$. Define $P(x)$ as follows

$$P(x) = \begin{cases} P_0(x) \text{ if } a \leq x \leq a + \dfrac{1}{2}, \\ 0 \qquad\ \text{ if } a + \dfrac{1}{2} < x \leq b - \dfrac{1}{2}, \\ P_1(x) \text{ if } b - \dfrac{1}{2} < x \leq b, \end{cases}$$

where P_i, $i = 0, 1$, are the third degree polynomials satisfying

$$(P_0(a), P_0'(a)) = y_0, \ (P_1(b), P_1'(b)) = y_1$$

and

$$(P_0(a + \frac{1}{2}), P_0'(a + \frac{1}{2})) = (P_1(b - \frac{1}{2}), P_1'(b - \frac{1}{2})) = (0, 0).$$

We may choose $\delta_0 > 0$ such that if $\|y_0\| \leq \delta \leq \delta_0$ and $\|y_1\| \leq \delta \leq \delta_0$, then $\|P\|_{L^\infty} \leq \eta$. It then follows from an easy computation that

$$\inf_{\mathcal{E}_{[a,b]}(\eta)} \mathcal{F}_g|_a^b \leq \mathcal{F}_g|_a^b(P) \leq C\delta^2, \tag{3.20}$$

where $C > 0$ essentially depends on $\|g\|_{L^\infty(-\eta,\eta)}$ and $\|f\|_{L^\infty(-\eta,\eta)}$.

Step 2 - Estimate of a minimizer. Assume that $\|y_0\| \le \delta \le \delta_0$ and $\|y_1\| \le \delta \le \delta_0$. The existence of a minimizer $u \in \mathcal{E}_{[a,b]}(\eta)$ follows by standard arguments. We deduce from the first step, by means of the inequality of Lemma 3.11, that

$$\|u\|_{H^2(a,b)} \le C\delta,$$

for some positive constant C. It then follows that

$$\|u\|_{C^1(a,b)} \le C\delta$$

and we infer from Lemma 3.12 that if $b - a \ge 1$, we may assume that C does not depend on the length of $[a, b]$. Hence, choosing δ_0 small enough, we conclude that

$$|u(x)| < \eta \quad \text{for all } x \in [a, b].$$

Therefore, u is in the interior of $\mathcal{E}_{[a,b]}(\eta)$ and solves (3.3) on $[a, b]$ together with the boundary conditions $(u(a), u'(a)) = y_0$ and $(u(b), u'(b)) = y_1$. The differential equation (3.3) then yields the estimate

$$\|u''''\|_{L^2} \le C\delta,$$

so that by interpolation we also have

$$\|u'''\|_{L^2} \le C\delta.$$

The constant C in the interpolation inequality can still be chosen independent of the length $b - a$ as long as $b - a \ge 1$, see R. A. Adams [**1**]. Now, the bound in C^3 follows from the bound in H^4 and the continuous injection in C^3. □

It then follows that the oscillatory behaviour of the solutions of the linearization (3.16) around the equilibrium extends to the minimizers of $\mathcal{F}_g|_a^b$ in $\mathcal{E}_{[a,b]}(\eta)$.

Lemma 3.14. *Suppose that the assumptions of Lemma* 3.13 *hold and let* $\eta > 0$ *be given by Lemma* 3.13. *Then there exist* $\delta_0 > 0$ *and* $\tau_0 > 0$ *such that if* $b - a \ge 1$, $\|y_0\| \le \delta_0$, $\|y_1\| \le \delta_0$, $\max(\|y_0\|, \|y_1\|) > 0$ *and* u *minimizes* $\mathcal{F}_g|_a^b$ *in* $\mathcal{E}_{[a,b]}(\eta)$, *u changes sign on every subinterval of* $[a, b]$ *having length larger than* τ_0.

Proof. A slight modification of the proof of Lemma 2.14 leads to the conclusion. It is easily seen that the assumption on the convergence of the solution to the equilibrium in Lemma 2.14 can be replaced by a condition ensuring that the solution remains close, for the C^3-norm, to 0. Here, thanks to the estimate of Lemma 3.13, the smallness of the boundary conditions suffices to obtain such a control. □

3.3.2. Existence of a Minimizer. We now focus on the proof of Theorem 3.10. Before turning to the proof itself, we check that, within our framework, the clipping process decreases the action.

LEMMA 3.15. *Assume the hypotheses of Theorem* 3.10 *hold. Then, if $\hat{u} \in \mathcal{E}$ is a clip of $u \in \mathcal{E}$, $\hat{u}$ has a smaller action than u.*

PROOF. Let $u \in \mathcal{E}$ and suppose the interval $[\alpha, \beta]$ may be clipped out. Then $u(\alpha) = u(\beta)$ and $u'(\alpha) = u'(\beta)$. It follows that the restriction of u to the interval $[\alpha, \beta]$ can be extended as a $\beta - \alpha$-periodic function. Then, arguing as in the proof of Theorem 3.7, we infer that $\mathcal{F}_g|_\alpha^\beta(u) \geq 0$. Here we can even prove that the last inequality is strict. This follows from assumption (C3) as

$$\inf_{\mathcal{E}} \mathcal{F}_{g-\frac{\varepsilon}{2}} > -\infty.$$

We now deduce that $\mathcal{F}_g(\hat{u}) < \mathcal{F}_g(u)$. $\square$

PROOF OF THEOREM 3.10. As $u = -1$ and $u = +1$ are minima of f, the assumptions imply the existence of $r \in]-1, 1[$ such that $f'(r) = 0$. Therefore, the constant function $u = r$ is an equilibrium state for (3.3). We may assume without loss of generality that $r = 0$.

Let $(u_n)_n \subset \mathcal{E}$ be a minimizing sequence for $\mathcal{F}_g$. We divide the proof in three parts. First, we settle some useful uniform bounds on the minimizing sequence. In the second part, we modified the sequence in a suitable way to avoid losses of compactness. In the last one, we focus on the convergence of the modified minimizing sequence.

PART 1 - A priori bounds.

Claim 1. We claim $(u_n')_n$ is uniformly bounded in $H^1(\mathbb{R})$ and

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} f(u_n)\, dx < +\infty.$$

Through the use of assumption (C3), we infer that

$$\frac{\varepsilon}{2}\int_{\mathbb{R}} {u_n'}^2\, dx \leq \int_{\mathbb{R}} (g(u_n) - \tilde{g}(u_n))\, \frac{{u_n'}^2}{2}\, dx = \mathcal{F}_g(u_n) - \mathcal{F}_{\tilde{g}}(u_n). \quad (3.21)$$

As the sequence $\mathcal{F}_g(u_n)$ is uniformly bounded and the infimum of $\mathcal{F}_{\tilde{g}}$ is finite, we deduce from the previous inequality that

$$\sup_{n\in\mathbb{N}} \|u_n'\|^2_{L^2(\mathbb{R})} < +\infty.$$

We now conclude that

$$\sup_{n\in\mathbb{N}} \left(\|u_n''\|^2_{L^2(\mathbb{R})}, \int_{\mathbb{R}} f(u_n)\, dx \right) \leq \sup_{n\in\mathbb{N}} \left(\mathcal{F}_g(u_n) + C\|{u_n'}^2\|^2_{L^2(\mathbb{R})} \right)$$

for some positive constant C which essentially depends on the lower bound on g. Hence, the claim follows.

Claim 2. The sequence $(u_n)_n$ is uniformly bounded in $C^1(\mathbb{R})$. This claim easily follows from the arguments of Part 1 in the proof of Proposition 2.9. Indeed, those arguments rely on a uniform bound on the sequences

$$\|u_n''\|_{L^2(\mathbb{R})} \quad \text{and} \quad \int_{\mathbb{R}} f(u_n)\,dx,$$

which is obtained in Claim 1.

We infer from Claim 1 and Claim 2 that there exists $M > 0$ such that

$$\sup_{n\in\mathbb{N}} \left(\|u_n'\|_{H^1(\mathbb{R})}, \int_{\mathbb{R}} f(u_n)\,dx \right) \le M.$$

Claim 3. If $(v_n)_n \subset \mathcal{E}$ is a minimizing sequence such that for all $n \in \mathbb{N}$, $\mathcal{F}_g(v_n) \le \mathcal{F}_g(u_n)$, then the sequence $(v_n)_n$ satisfies the same uniform bounds as $(u_n)_n$, i.e.

$$\sup_{n\in\mathbb{N}} \left(\|v_n'\|_{H^1(\mathbb{R})}, \int_{\mathbb{R}} f(v_n)\,dx \right) \le M.$$

We deduce as in estimate (3.21) that

$$\begin{aligned} \frac{\varepsilon}{2} \int_{\mathbb{R}} {v_n'}^2\,dx &\le \mathcal{F}_g(v_n) - \mathcal{F}_{\tilde{g}}(v_n) \\ &\le \mathcal{F}_g(u_n) - \mathcal{F}_{\tilde{g}}(v_n). \end{aligned}$$

Therefore, the uniform bound for $\|v_n'\|_{L^2(\mathbb{R})}$ does only depend on the bound for $\mathcal{F}_g(u_n)$ and the infimum of $\mathcal{F}_{\tilde{g}}$ in $\mathcal{E}$. As similar observations can be made for the other bounds, the claim follows.

Part 2 - Choice of a particular minimizing sequence. Let x_1 and x_2 denote respectively the first and the last zero of u_n. We then define the space

$$\mathcal{E}^n_{[x_1,x_2]} := \{u \in \mathcal{E} \mid u = u_n \text{ in } \mathbb{R}\backslash\,]x_1, x_2[\}.$$

Since $\mathcal{F}_g$ is bounded from below in $\mathcal{E}$, the same holds true for $\mathcal{F}_g|_{x_1}^{x_2}$ in $\mathcal{E}^n_{[x_1,x_2]}$. We therefore deduce from standard arguments that there exists $v_n \in \mathcal{E}^n_{[x_1,x_2]}$ such that

$$\mathcal{F}_g|_{x_1}^{x_2}(v_n) = \inf_{\mathcal{E}^n_{[x_1,x_2]}} \mathcal{F}_g|_{x_1}^{x_2}.$$

Moreover, v_n solves equation (3.3) in $[x_1, x_2]$ together with the boundary conditions $v_n(x_1) = v_n(x_2) = 0$ and $v_n'(x_1) = u_n'(x_1)$, $v_n'(x_2) = u_n'(x_2)$. If $u_n'(x_1) = u_n'(x_2)$, then the interval $[x_1, x_2]$ may be clipped out and this part of the proof can be skipped. In the sequel we therefore assume $u_n'(x_1) \neq u_n'(x_2)$.

We first observe that the zeros of v_n are isolated. Let Z denote the set of zeros of v_n. The claim easily follows from the fact that $u = 0$ is an equilibrium state. Indeed, as v_n is of class C^4, a straightforward repeated use of Rolle's theorem implies that if Z has a cluster point

$x_0 \in Z$, then $v_n(x_0) = v_n'(x_0) = v_n''(x_0) = v'''(x_0)$. But in this case, the uniqueness of the solution of the Cauchy problem, with initial conditions taken at point x_0, implies $v_n(x) = 0$ for all $x \in [x_1, x_2]$. This contradicts the fact that $v_n'(x_1) \neq v_n'(x_2)$.

As the zeros of v_n are isolated, the interval $[x_1, x_2]$ may be cut into a finite number of subintervals whose extremities are two successive zeros. We define two types of intervals. Let $\delta_0 > 0$ be given by Lemma 3.14. We say that $I \subset [x_1, x_2]$ is of type A if

$$\max_{x \in I} |v_n(x)| \leq 1 - \delta_0 \qquad \text{(type A)},$$

while a subinterval $I \subset [x_1, x_2]$ on which

$$\max_{x \in I} |v_n(x)| > 1 - \delta_0 \qquad \text{(type B)},$$

will be referred to as an interval of type B. In the next three steps, we prove that the interval $[x_1, x_2]$ is uniformly bounded. In step 1, we obtain an upper bound on the length of the union of type A intervals. We then prove in step 2 that the length of any interval of type B is uniformly bounded. We finally show that the number of type B intervals is uniformly bounded as well.

Step 1 - Estimate of the length of type A intervals. Let us denote by $\mathcal{A}$ the union of all the intervals of type A and define

$$\mu := \min\{f(u) \mid |u-1| > \delta_0 \text{ and } |u+1| > \delta_0\}.$$

We then obtain the estimate

$$\mu|\mathcal{A}| \leq \int_{\mathcal{A}} f(v_n)\,dx \leq \sup_{n \in \mathbb{N}} \int_{\mathbb{R}} f(v_n)\,dx.$$

Step 2 - Estimate of the length of type B intervals. Let $I := [x_i, x_{i+1}]$ be an interval of type B and assume to fix the ideas that v_n is positive in the interior of I.

Claim 1. We may assume that v_n takes at most two times each of the values $1-\delta_0$ and $1+\delta_0$. Let $c \in I$ be such that

$$\max_{x \in I} v_n(x) = v_n(c) > 1 - \delta_0.$$

The idea is to use the clipping procedure to discard the possible oscillations of v_n around $1-\delta_0$ and $1+\delta_0$. We infer from Lemma 3.15 that the clipping process decreases the action. Suppose that v_n crosses $1-\delta_0$ more than once at the left of c. We then define

$$\begin{aligned}
\xi_2 &= \min\{x \in [x_i, c[\mid v_n'(x) = 0 \text{ and } v_n(x) \geq 1 - \delta_0\},\\
\xi_4 &= \max\{x \in [\xi_2, c] \mid v_n(x) = v_n(\xi_2)\},\\
\xi_3 &= \max\{x \in [\xi_2, \xi_4] \mid v_n'(x) = 0\}
\end{aligned}$$

and

$$\xi_1 = \max\{x \in [x_i, \xi_2] \mid v_n(x) = v_n(\xi_3)\}.$$

We are now in a position to apply the clipping procedure of Lemma 2.12. Hence, we find $s_1 \in [\xi_1, \xi_2]$ and $s_2 \in [\xi_3, \xi_4]$ such that $v_n(s_1) = v_n(s_2)$ and $v_n'(s_1) = v_n'(s_2)$. It follows that the corresponding clip $\hat{v}_n$ satisfies $\mathcal{F}_g(\hat{v}_n) \le \mathcal{F}_g(v_n)$ and since $[\xi_2, \xi_3]$ is contained in $[s_1, s_2]$, the restriction of $\hat{v}_n$ to $[x_i, c-(s_2-s_1)]$ takes the value $1-\delta_0$ only once at the left of $c-(s_2-s_1)$.

As the same kind of modifications can be done at the right of c, we obtain a function that takes at most two times the value $1-\delta_0$. In particular, we can repeat the arguments to discard the possible oscillations around $1+\delta_0$ so that the claim follows.

We now subdivide the interval I into five parts. We denote by I_1 and I_5 the intervals where v_n is below $1-\delta_0$ and containing respectively the left and the right-hand side of I. We call I_3 the part of I where v_n is above $1+\delta_0$. The intervals I_2 and I_4 denote the subintervals where $1-\delta_0 \le v_n(x) \le 1+\delta_0$ respectively at the left and at the right of I_3. Clearly, if I_3 is empty, we only consider three subintervals and we denote the middle interval by $\bar{I}_3$. Arguing as in Step 1, we infer that the length of $I_1 \cup I_3 \cup I_5$ is uniformly bounded.

We next consider three different cases. The parameter $\eta > 0$ is as in Lemma 3.13, i.e. such that there exist $\beta > 0$ and $k > 0$ satisfying $\beta < \sqrt{8k}$ such that for all $u \in [1-\eta, 1+\eta]$,

$$g(u) \ge -\beta \quad \text{and} \quad f(u) \ge k(u-1)^2 \tag{3.22}$$

and for all $u \in [-1-\eta, -1+\eta]$,

$$g(u) \ge -\beta \quad \text{and} \quad f(u) \ge k(u+1)^2. \tag{3.23}$$

It follows from Lemma 3.13 that we may assume $0 < \delta_0 < \eta$.

Case 1: the interval I_3 is non-empty and

$$\max_{x \in I} v_n(x) \ge 1+\eta.$$

Case 2: the interval I_3 is non-empty and

$$\max_{x \in I} v_n(x) < 1+\eta.$$

Case 3: the interval I_3 is empty.

Let $\tau_0 > 0$ be defined by Lemma 3.14.

Claim 2. If Case 1 holds, we may assume that $|I_i| \le 4 + \max(1, 4\tau_0)$ *for* $i = 2, 4$. We focus on the interval $I_2 := [\theta_1, \theta_2]$. We then define

$$a = \inf\{x \ge \theta_1 \mid |u(x) - 1| \le \delta_0 \text{ and } |u'(x)| \le \delta_0\}$$

and

$$b = \sup\{x \le \theta_2 \mid |u(x) - 1| \le \delta_0 \text{ and } |u'(x)| \le \delta_0\}.$$

Notice that a and b need not exist. In the case where they do not, we have $|I_2| \le 2$. Indeed, the variation of v_n is equal to $2\delta_0$ and $v_n'(x) \ge \delta_0$ so that

$$2\delta_0 = \int_{\theta_1}^{\theta_2} v_n'(x)\,dx \ge \delta_0(\theta_2 - \theta_1).$$

Suppose now that a and b do exist and assume $b - a \ge \max(1, 4\tau_0)$. Let $\mathcal{E}_{[a,b]}(\eta)$ be the space defined by (3.18), with $y_0 = (v_n(a) - 1, v_n'(a))$ and $y_1 = (v_n(b) - 1, v_n'(b))$. As the restriction of $v_n - 1$ to the interval $[a, b]$ is in $\mathcal{E}_{[a,b]}(\eta)$, we may assume v_n minimizes $\mathcal{F}_g|_a^b$ in $\mathcal{E}_{[a,b]}(\eta) + 1$ otherwise we locally modify v_n in $[a, b]$. Since $\|y_i\| \le \delta_0$ for $i = 0, 1$, we infer from the analysis of Lemma 3.14 that v_n oscillates around $+1$ in each subinterval of length greater than τ_0. We then define

$$a' = \max\{x \le a \mid v_n(x) = 1 - \eta\}$$

and

$$b' = \min\{x \ge b \mid v_n(x) = 1 + \eta\},$$

so that $|v_n(x) - 1| \le \eta$ for all $x \in [a', b']$. Following by now familiar arguments, we define

$$\begin{aligned}\xi_2 &= \min\{x \in [a', b'] \mid v_n'(x) = 0 \text{ and } v_n(x) \ge 1\},\\ \xi_4 &= \max\{x \in [a', b'] \mid v_n(x) = v_n(\xi_2)\},\\ \xi_3 &= \max\{x \in [a', b'] \mid v_n'(x) = 0\}\end{aligned}$$

and take

$$\xi_1 = \max\{x \in [a', \xi_2] \mid v_n(x) = v_n(\xi_3)\}.$$

Observe that Lemma 3.14 ensures the existence of $\xi_2 \in [a', a + 2\tau_0]$. Also, we have $\xi_3 \in [a - 2\tau_0, b']$. We are now able to apply the clipping procedure. Consequently, we can discard the restriction of v_n to some interval $[s_2, s_3]$ containing $[\xi_2, \xi_3]$. Letting $\theta_2^* = \theta_2 - (s_3 - s_2)$, we have

$$\theta_2^* - \theta_1 \le (a - \theta_1) + 4\tau_0 + (\theta_2 - b).$$

Arguing as above, we infer that $a - \theta_1 \le 2$ and $\theta_2 - b \le 2$ so that we obtain the desired estimate.

Claim 3. If Case 2 holds, we may assume that

$$|I_2 \cup I_3 \cup I_4| \le 6 + \max(1, 8\tau_0) + |I_3|.$$

Let $I_2 = [\theta_1, \theta_2]$ and $I_4 = [\theta_3, \theta_4]$. We redefine

$$a = \inf\{x \ge \theta_1 \mid |u(x) - 1| \le \delta_0 \text{ and } |u'(x)| \le \delta_0\}$$

and

$$b = \sup\{x \le \theta_4 \mid |u(x) - 1| \le \delta_0 \text{ and } |u'(x)| \le \delta_0\}.$$

If a and b exist, we have three possibilities: (i) $\theta_1 \le a \le b \le \theta_2 \le \theta_3 < \theta_4$, (ii) $\theta_1 \le a \le \theta_2 \le \theta_3 \le b \le \theta_4$ or (iii) $\theta_1 < \theta_2 \le \theta_3 \le a \le b \le \theta_4$. If case (i) holds, we have $a - \theta_1 \le 2$, $\theta_2 - b \le 2$ and $\theta_4 - \theta_3 \le 2$. In case (ii), we have $a - \theta_1 \le 2$ and $\theta_4 - b \le 2$ while in case (iii), we have $\theta_2 - \theta_1 \le 2$,

$a - \theta_3 \leq 2$ and $\theta_4 - b \leq 2$. Assume that $b - a \geq \max(1, 8\tau_0)$. As in Claim 2, we may assume that v_n minimizes $\mathcal{F}_g|_a^b$ in the space $\mathcal{E}_{[a,b]}(\eta)$. We then define

$$a' = \max\{x \leq a \mid v_n(x) = 1 - \eta\}$$

and

$$b' = \min\{x \geq b \mid v_n(x) = 1 - \eta\}.$$

Let $q \in [a', b']$ be such that $\max_{x \in [a',b']} v_n(x) = v_n(q)$. We deduce from Lemma 3.14 that $v_n(q) > 1$ so that we may apply the arguments used in the preceding claim to each of the intervals $[a', q]$ and $[q, b']$. If we are either in situation (i) or (iii) described above, we apply the clipping process in $[a', q]$ if $q \in [\theta_2, \theta_3]$ or in both $[a', q]$ and $[q, b']$ in the contrary. Denoting by θ_4^* the point into which θ_4 is transformed, we have after clipping, either

$$\theta_4^* - \theta_1 \leq 6 + 4\tau_0 + |I_3|$$

or

$$\theta_4^* - \theta_1 \leq 4 + 6\tau_0 + |I_3|.$$

If case (ii) holds, then we apply the clipping process in both $[a', q]$ and $[q, b']$ and we conclude that

$$\theta_4^* - \theta_1 \leq 4 + 8\tau_0.$$

Hence, the claim is proved.

Claim 4. If Case 3 holds, we may assume that $|\bar{I}_3| \leq 4 + 8\tau_0$. The proof of this claim follows the lines of that of Claim 3.

Conclusion of Step 2. We deduce from the previous claims that the length of an interval of type B is uniformly bounded.

Step 3 - Estimate of the number of type B intervals. Let m denote the number of type B intervals. We deduce from the first part of the proof that the sequence $(v_n)_n$ is uniformly bounded in $C^1(\mathbb{R})$. Let $L > 0$ denote a bound on the derivative. Let I be an interval of type B on which v_n is non-negative. Keeping the notations introduced in Step 2, we have $|v_n(x)| \leq 1 - \delta_0$ for $x \in I_1 \cup I_5$. We then compute

$$1 - \delta_0 = \int_{I_1} v_n'(x)\, dx \leq |I_1| L$$

so that

$$|I_1| \geq \frac{1 - \delta_0}{L}.$$

Since the same estimate holds for $|I_5|$, we come out with the inequality

$$\int_I f(v_n(x))\, dx \geq \int_{I_1 \cup I_5} f(v_n(x))\, dx \geq \frac{2\mu(1 - \delta_0)}{L}.$$

The same lower bound is valid when I is an interval of type B on which v_n is non-positive. We now conclude that

$$\int_{\mathbb{R}} f(v_n(x))\,dx \geq m\frac{2\mu(1-\delta_0)}{L}$$

and since the left-hand side of the inequality is uniformly bounded, the same holds true for m.

Conclusion of Part 2. It follows from the previous steps that we may choose a minimizing sequence $(v_n)_n \subset \mathcal{E}$ such that for some uniformly bounded interval $[x_1^n, x_2^n] \subset \mathbb{R}$,

$$v_n(x) < 0 \text{ for } x < x_1^n \text{ and } v_n(x) > 0 \text{ for } x > x_2^n.$$

By translation invariance, we may assume that the first zero of v_n is achieved at $x_1^n = 0$.

PART 3 - Convergence. We deduce from the first and second parts of the proof that, up to a subsequence, $x_2^n \to x_2$

$$v_n \xrightarrow{C^1_{\text{loc}}(\mathbb{R})} v \text{ and } v_n' \overset{H^1(\mathbb{R})}{\rightharpoonup} v',$$

for some $v \in H^2_{\text{loc}}(\mathbb{R})$. Therefore, $v(x) \leq 0$ for $x \leq 0$ and $v(x) \geq 0$ for $x \geq x_2$. On the other hand, we infer that

$$\int_{\mathbb{R}} f(v(x))\,dx < +\infty.$$

Indeed, for any compact interval $[-T, T]$, we have

$$\int_{-T}^{T} f(v(x))\,dx = \lim_{n\to+\infty} \int_{-T}^{T} f(v_n(x))\,dx \leq \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} f(v_n(x))\,dx.$$

Now, arguing as in the proof of Theorem 2.8, we conclude that

$$\lim_{x\to\pm\infty} v(x) = \pm 1.$$

It then follows, using the quadraticity of the potential around the equilibria, that v do belong to $\mathcal{E}$. To see that v is a minimizer, we combine the uniform convergence in compact intervals and the arguments considered at the end of the proof of Theorem 3.6. First, observe that arguing as in Part 2, we may assume there exists $T > 0$ such that for every $n \in \mathbb{N}$, $|v_n(x)+1| \leq \eta$ for $x < -T$ and $|v_n(x)-1| \leq \eta$ for $x > T$. By means of (3.22) and (3.23), we actually deduce that (up to a subsequence)

$$v_n + 1 \overset{H^2(-\infty,-T)}{\rightharpoonup} v + 1 \text{ and } v_n - 1 \overset{H^2(T,+\infty)}{\rightharpoonup} v - 1.$$

Let $J_1 =]-\infty, -T[$, $J_2 = [-T, T]$ and $J_3 =]T, +\infty[$. On the one hand, it is easy to check that

$$\mathcal{F}_g|_{J_2}(v) \leq \liminf_{n\to+\infty} \mathcal{F}_g|_{J_2}(v_n). \tag{3.24}$$

On the other hand, in J_1, we know that $|v_n(x)+1| \le \eta$. We therefore write

$$\begin{aligned}\mathcal{F}_g|_{J_1}(v_n) &= \frac{1}{2}\int_{J_1}\left(v_n''^2 - \beta v_n'^2 + \frac{\beta^2}{4}(v_n-1)^2\right)dx \\ &\quad + \int_{J_1}\frac{g(v_n)+\beta}{2}\,v_n'^2\,dx + \int_{J_1}\left(f(v_n) - \frac{\beta^2}{8}(v_n-1)^2\right)dx \\ &= \frac{1}{2}\left(\int_{J_1}\left(v_n'' + \frac{\beta}{2}(v_n-1)\right)^2 dx + \beta v_n'(-T)v_n(-T)\right) \\ &\quad + \int_{J_1}\frac{g(v_n)+\beta}{2}v_n'^2\,dx + \int_{J_1}\left(f(v_n) - \frac{\beta^2}{8}(v_n-1)^2\right)dx.\end{aligned}$$

In the last equality, the first integral is convex and Fatou's Lemma is applicable to the last two so that we get the inequality

$$\mathcal{F}_g|_{J_1}(v) \le \liminf_{n\to+\infty}\mathcal{F}_g|_{J_1}(v_n).$$

Since the contribution on the interval J_3 can be handled in the same way and taking also (3.24) into account, we deduce that

$$\mathcal{F}_g(v) \le \liminf_{n\to+\infty}\mathcal{F}_g(v_n) = \inf_{\mathcal{E}}\mathcal{F}_g,$$

which implies v is a minimizer. Finally, the fact that v is a classical heteroclinic solution follows from by now familiar arguments. □

The analysis of Section 2.3 concerning the qualitative properties of the minimizer applies again to the minimizer obtained in Theorem 3.10. Indeed, we have seen in Lemma 3.15 that the clipping process still decreases the action.

3.4. Notes and Comments

NOTE 3.1. A remark similar to that in Note 2.2 may be formulated concerning the minimizer obtained in Theorem 3.3. In particular, if ± 1 are nondegenerate minima of f, the minimizer in $\mathcal{H}$ is a classical heteroclinic connection between -1 and $+1$.

NOTE 3.2. A sharp estimate of the critical parameter β_0 introduced in Theorem 3.9 is still missing. A very rough estimate can be obtained via Lemma 3.5 and an estimate of the best constant for the interpolation inequality used therein. However, this estimate is far from the numerical observation of J. B. van den Berg (see Conjecture 7 in [**112**]) who claims $\beta_0 \approx -0.92$.

NOTE 3.3. Let us define $\mathcal{C}^+$ as the cone of positive functions in $\mathcal{E}^+$, i.e.

$$\mathcal{C}^+ := \{u \in \mathcal{E}^+ \mid u(x) > 0 \text{ for } x > 0\}$$

and $\mathcal{F}^+_\beta : \mathcal{E}^+ \to \mathbb{R}$ the restricted functional

$$\mathcal{F}^+_\beta(u) = \int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2-1)^2 \right) dx.$$

It follows from Theorem 3.9 that $\mathcal{F}^+_\beta$ has a minimizer in $\mathcal{E}^+$ for every $\beta \geq \beta_0$. Moreover, this minimizer belongs to the interior of $\mathcal{C}^+$. Indeed, as previously emphasized, the Hamiltonian

$$H(u,\, u',\, u'',\, u''') = u'''u' - \frac{1}{2}u''^2 - \frac{\beta}{2}u'^2 + \frac{(u^2-1)^2}{4}$$

is constant along solutions. As for any minimizer $\varphi \in \mathcal{E}$, we have $H(\varphi,\, \varphi',\, \varphi'',\, \varphi''') = 0$ and $\varphi''(0) = 0$, we deduce from the conservation of the Hamiltonian that $\varphi'(0) > 0$. On the other hand, we deduce from Step 2 of the proof of Theorem 3.6 that

$$\inf_{\mathcal{C}^+} \mathcal{F}^+_\beta > -\infty$$

for every $\beta > -\sqrt{2}$. Hence it is natural to ask if $\mathcal{F}^+_\beta$ has a local minimizer in the interior of $\mathcal{C}^+$ for $-\sqrt{2} < \beta \leq \beta_0$. To our knowledge, this question remains open. However, we can observe that an argument of continuity implies that the answer is positive at least for β close to β_0. Indeed, it is easily seen that $\mathcal{F}^+_\beta$ has a local minimizer in $\mathcal{C}^+$ for every $\beta > -\sqrt{2}$. For $\beta \geq \beta_0$, we can prove that the minimizers cannot lie on the boundary of $\mathcal{C}^+$ so that

$$\mathcal{F}^+_{\beta_0}(\varphi_{\beta_0}) < \inf_{\partial\mathcal{C}^+} \mathcal{F}^+_{\beta_0}, \tag{3.25}$$

where φ_{β_0} is a minimizer of $\mathcal{F}^+_{\beta_0}$. On the other hand, for $\beta_1 > -\sqrt{2}$ and $\beta \in [\beta_1, \beta_0]$, we obtain

$$\mathcal{F}^+_\beta(u) = \mathcal{F}^+_{\beta_1}(u) + \frac{\beta-\beta_1}{2}\int_{\mathbb{R}^+} u'^2\,dx \geq -C + \frac{\beta-\beta_1}{2}\int_{\mathbb{R}^+} u'^2\,dx.$$

We now infer from this last estimate that any function $u \in \mathcal{C}^+$ satisfying

$$\mathcal{F}^+_\beta(u) = \inf_{\partial\mathcal{C}^+} \mathcal{F}^+_\beta$$

is a priori bounded in $\mathcal{E}^+$ for β in any compact subinterval of $]\beta_1, \beta_0]$. It follows that the inequality (3.25) holds true for β in a left neighborhood of β_0, i.e.

$$\mathcal{F}^+_\beta(\varphi_{\beta_0}) < \inf_{\partial\mathcal{C}^+} \mathcal{F}^+_\beta.$$

Since φ_{β_0} is in the interior of $\mathcal{C}^+$, this means that the infimum of $\mathcal{F}^+_\beta$ in $\mathcal{C}^+$ cannot be achieved on the boundary.

Note 3.4. It is worth pointing out that the assumption

$$-2\sqrt{f''(\pm 1)} < g(\pm 1)$$

in Theorem 3.10 is necessary to reach the conclusion. Indeed, if f is purely quadratic around ± 1, e.g. $f(u) = (u \pm 1)^2$ for $|x \pm 1| \leq \delta$, where $\delta > 0$, it is easily seen that heteroclinics for the equation

$$u'''' - \beta u'' + f'(u) = 0$$

cannot exist for $\beta \leq -\sqrt{8}$.

On the other hand, the fact that ε has to be taken strictly positive in assumption (C3) seems to be of a technical nature. However, we do not know if the result remains valid without the full power of this assumption.

Chapter 4

Multi-transition Connections

This chapter is devoted to the multiplicity of heteroclinic and homoclinic connections when the equilibria are of saddle-focus type. We first briefly discuss the results of W. D. Kalies, J. Kwapisz and R. C. A. M. VanderVorst [**54**]. We then show how we extended their method in D. Bonheure [**15**].

4.1. Homotopy Classes of Heteroclinic Solutions

When $-\sqrt{8} < \beta < \sqrt{8}$, the constant solutions $u = \pm 1$ of the model equation

$$u'''' - \beta u'' + u^3 - u = 0 \tag{4.1}$$

are saddle-focus equilibria. As underlined in the introduction, it is known that the set of heteroclinic connections between such kind of equilibria can be extremely complex [**21, 26, 41**]. L. A. Peletier and W. C. Troy [**79**] were the first to obtain multiplicity results for equation (4.1) by means of a topological shooting method. In [**55**], W. D. Kalies and R. C. A. M. VanderVorst constructed multi-bump solutions while W. D. Kalies, J. Kwapisz and R. C. A. M. VanderVorst [**54**] found a fairly rich set of multi-transition solutions. All these results concern the parameter range $0 \le \beta < \sqrt{8}$. The method of W. D. Kalies, J. Kwapisz and R. C. A. M. VanderVorst consists in minimizing the action functional $\mathcal{F}_\beta : \mathcal{E} \to \mathbb{R}$, where we recall that

$$\mathcal{F}_\beta(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2 - 1)^2 \right) dx \tag{4.2}$$

and

$$\mathcal{E} = \{u : \mathbb{R} \to \mathbb{R} \mid u(0) = 0,\ u + 1 \in H^2(\mathbb{R}^-),\ u - 1 \in H^2(\mathbb{R}^+)\}, \tag{4.3}$$

in specific homotopy classes of functions. When projected in the configuration plane (u, u'), the trajectory of a heteroclinic solution yields a curve connecting the points $(\pm 1, 0)$. A winding vector that records the number of turns around $(-1, 0)$ and $(+1, 0)$ can then be associated to any curve connecting $(\pm 1, 0)$, see the example depicted in Figure 7 in the introduction. For each vector $\omega \in 2\mathbb{N}_0^{2m} \cup \{0\}$, we may define the homotopy class $M(\omega)$ consisting of functions of $\mathcal{E}$ having winding vector $\omega/2$. The components of the vector ω contain the exact number of crossings with either -1 or $+1$ between the transitions. The following more precise definition was introduced in W. D. Kalies et al. [**54**]. In this statement, we denote by $u^{-1}(\pm 1)$ the set $\{x \in \mathbb{R} \mid u(x) = -1 \text{ or } u(x) = +1\}$.

Definition 4.1. *Assume* $\omega \in 2\mathbb{N}_0^{2m} \cup \{0\}$. *If* ω *is the vector* 0, *we set* $m = 0$. *A function* $u \in \mathcal{E}$ *is in the class* $M(\omega)$ *if there are nonempty sets* $A_0, \ldots, A_{2m+1}$ *such that*
(i) $u^{-1}(\pm 1) = A_0 \cup \ldots \cup A_{2m+1}$,
(ii) $\# A_i = \omega_i$ *for* $i = 1, \ldots, 2m$,
(iii) $\max A_i < \min A_{i+1}$ *for* $i = 0, \ldots, 2m$,
(iv) $u(A_i) = (-1)^{i+1}$,
(v) $\{\max A_0\} \cup A_1 \cup \ldots \cup A_{2m} \cup \{\min A_{2m+1}\}$ *consists of transverse crossings of* ± 1.

Under these conditions $M(\omega)$ is an open subset of $\mathcal{E}$. As the functional $\mathcal{F}_\beta$ is positive, we may define, for each vector $\omega \in 2\mathbb{N}^{2m} \cup \{0\}$, the value

$$\inf_{u \in M(\omega)} \mathcal{F}_\beta(u). \tag{4.4}$$

If the infimum is achieved by a function in the interior of $M(\omega)$, then a local minimizer of $\mathcal{F}_\beta$ with the corresponding properties solves the Euler-Lagrange equation (4.1). The next theorem is a special case of Theorem 1.3 in [**54**].

Theorem 4.2. *Let* $0 < \beta < \sqrt{8}$. *Then, for every* $\omega \in 2\mathbb{N}^{2m} \cup \{0\}$, *the functional* $\mathcal{F}_\beta : \mathcal{E} \to \mathbb{R}$ *defined by* (4.2) *and* (4.3), *has a local minimizer* u_ω *in the homotopy class* $M(\omega)$. *In particular, the function* u_ω *is a heteroclinic solution of* (4.1) *connecting* -1 *to* $+1$ *and displaying* $2m + 1$ *transitions between* ± 1.

The proof of Theorem 4.2 is carried out in [**54**]. The main difficulty when looking for local minimizers in homotopy classes, is to check that the minimizing sequence does not converge to a function which lies on the boundary of the class. As emphasized in the introduction, the limit function could either have lost or gain complexity, due to the coalescence of two or more crossings or to the apparition of spurious oscillations. We do not enter into the detail of the proof of Theorem 4.2. However, some of the arguments developed by W. D. Kalies et al. are discussed in the next section and adapted to a more general framework.

4.2. Multi-transition Heteroclinics

In this section, we prove the existence of multi-transition heteroclinic solutions of (4.1) for $\beta \geq \beta_0$, where β_0 is defined by (3.13). By a transition, we mean a jump from -1 to 1 or reversely. Actually, we consider the more general functional

$$\mathcal{F}_g(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \tag{4.5}$$

whose Euler-Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{4.6}$$

We assume that f and g are even and the equilibrium states $u = \pm 1$ are of saddle-focus type. Since the functional is symmetric, we may restrict our attention to odd solutions. Indeed, as already asserted in Chapter 2, given any function $u \in \mathcal{E}$, it is easy to build an odd function $u^* \in \mathcal{E}$ having smaller action than u. Hence, we may work out the minimization process with minimizing sequence composed of odd functions. We then look at the minimizers of the functional

$$\mathcal{F}_g^+(u) = \int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \tag{4.7}$$

defined in the space

$$\mathcal{E}^+ = \{u \mid u - 1 \in H^2(\mathbb{R}^+),\ u(0) = 0\}. \tag{4.8}$$

If $\mathcal{F}_g^+$ has a minimizer u, an easy computation (see the proof of Theorem 2.3) shows that $u''(0) = 0$. Extending then u to $\mathbb{R}$ by

$$u^*(x) = \begin{cases} -u(-x) & \text{if } x < 0, \\ u(x) & \text{if } x \geq 0, \end{cases} \tag{4.9}$$

we obtain an odd solution of (4.6).

We search for multi-transition heteroclinic solutions as local minima of the functional $\mathcal{F}_g$ in appropriate subsets of $\mathcal{E}$. Basically, these subsets correspond to classes of functions having the desired number of transitions. With regards to the result of W. D. Kalies et al., the multi-transition solutions we look for by now, correspond to minimizers in homotopy classes $M(\omega)$ with winding vectors $\omega/2$ having all its components equal to one.

We define for each $p \geq 0$, the subset $\mathcal{E}_p^+ \subset \mathcal{E}^+$ consisting of functions whose odd extensions to $\mathbb{R}$ make $2p + 1$ transitions.

Definition 4.3. *A function $u \in \mathcal{E}^+$ belongs to the subclass $\mathcal{E}_p^+$ if there exist $0 = x_0 < x_1 < \ldots < x_p < x_{p+1} = +\infty$ such that*

$$u(x)(-1)^{i+p} > 0 \text{ for } x \in I_i :=]x_i, x_{i+1}[$$

and

$$\max_{x\in I_i} u(x)(-1)^{i+p} > 1,$$

for every $i = 0, \dots, p$.

4.2.1. Functionals of Swift-Hohenberg Type. We first consider the setting of Section 3.2. Namely, we assume that f and $g \in C^2(\mathbb{R})$ are even functions such that $f(1) = 0$ and (C2) holds, i.e. for some $k > 0$, $\beta \in [0, \sqrt{8k})$ and all $u \geq 0$,

$$f(u) \geq k(u-1)^2 \quad \text{and} \quad g(u) \geq -\beta.$$

We prove that the functional $\mathcal{F}_g^+$ has a local minimum in each subspace $\mathcal{E}_p^+$ in the following situation.

THEOREM 4.4. *Let* f, $g \in C^2(\mathbb{R})$ *be even functions such that* (C2) *holds and* $f(1) = 0$. *Assume moreover that* $g(1)^2 < 4f''(1)$ *and*

$$\inf_{\mathcal{E}^+} \mathcal{F}_g^+ > -\infty,$$

where the functional $\mathcal{F}_g^+ : \mathcal{E}^+ \to \mathbb{R}$ *is defined by* (4.7) *and* (4.8). *Then, for every* $p \in \mathbb{N}$, *there exists a local minimizer of* $\mathcal{F}_g^+$ *in the subspace* $\mathcal{E}_p^+$ *described in Definition* 4.3. *Moreover, its odd extension to* $\mathbb{R}$ *is a heteroclinic solution of* (4.6) *connecting* -1 *to* $+1$ *and having exactly* $2p+1$ *zeros.*

The case of a non-negative function g is already covered by the contribution of W. D. Kalies et al. [**54**] where even more precise results are given in the spirit of that in Theorem 4.2. However, our result can be seen as a partial extension of the methods of [**54**] to Lagrangians that can take either sign. To deal with such Lagrangians, we only require an a priori lower bound on the action along admissible functions.

The main idea in the proof of Theorems 4.4 is to show the existence of a minimizing sequence $(u_n)_n \subset \mathcal{E}_p^+$ that has the following properties:

(a) there exists $I > 0$ such that for all $n \in \mathbb{N}$, the zeros of u_n are such that $|I_i| \leq I$ for all $i = 0, \dots, p-1$;
(b) there exists $C > 0$ such that for all u_n, $\|u_n - 1\|_{H^2(\mathbb{R}^+)} \leq C$.

These two properties are closely related as they overcome a lack of compactness when extracting a weak converging subsequence.

The main tool for obtaining the estimates on the length of the intervals I_i is the clipping procedure described in Section 2.3.1. Also, the oscillatory behaviour of local minimizers close to the equilibria ± 1, described in Section 3.3.1, is crucial in the construction of a minimizing sequence having the above properties.

Estimates on minimizing sequences. We recall that $u \in \mathcal{E}_p^+$ if there exist $0 = x_0 < x_1 < \ldots < x_p < x_{p+1} = +\infty$ such that

$$u(x)(-1)^{i+p} > 0 \text{ for } x \in I_i,$$

and

$$\max_{x \in I_i} u(x)(-1)^{i+p} > 1,$$

where we still denote by I_i the intervals $]x_i, x_{i+1}[$ for $i = 0, \ldots, p$. To avoid some confusions, we sometimes use the notation I_i^u to emphasize that these intervals correspond to the transitions of a given function u. We still use the notation

$$\mathcal{F}_g|_{I_i}(u) = \int_{x_i}^{x_{i+1}} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx.$$

The aim of this section is to construct a minimizing sequence $(u_n)_n$ that satisfies properties (a) and (b) described above.

We first show that under the assumption (C2), the clipping process still reduces the action when the function is of one sign on the discarded pieces.

Lemma 4.5. *Let f and $g \in C(\mathbb{R})$ be even functions such that* (C2) *holds. Let $u \in H^2(a, b)$ be such that $u(a) = u(b)$ and $u'(a) = u'(b)$. Assume moreover that u is of one sign. Then*

$$\int_a^b \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \geq 0.$$

Proof. Suppose that u is non-negative in $]a, b[$. Then using the inequality of Lemma 3.11 applied to $u - 1$, we obtain

$$\begin{aligned}\mathcal{F}_g|_a^b(u) &\geq \int_a^b \left(\frac{1}{2}(u''^2 - \beta u'^2) + k(u-1)^2 \right) dx \\ &\geq \varepsilon \|u - 1\|_{H^2(a,b)}^2.\end{aligned}$$

If u is non-positive in $]a, b[$, we conclude that

$$\int_a^b \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \geq \varepsilon \|u + 1\|_{H^2(a,b)}^2.$$

□

Our first step in the construction of a minimizing sequence having properties (a) and (b), is to look for a lower bound for $\mathcal{F}_g|_{I_i}(u)$ for all $i = 0, \ldots, p$. As a consequence, we obtain a lower bound for $\mathcal{F}_g^+$ in $\mathcal{E}_p^+$ for each $p \in \mathbb{N}$.

Lemma 4.6. *Let $M > 0$ and assume that $f, g \in C(\mathbb{R})$ are even functions satisfying assumption* (C2). *Then, there exists $K > 0$ such that if $u \in \mathcal{E}_p^+$ satisfies $\mathcal{F}_g^+(u) < M$, we have $\mathcal{F}_g|_{I_i}(u) \geq -K$ for all $i = 0, \ldots, p$.*

Proof. Let u be a function of $\mathcal{E}_p^+$ such that $\mathcal{F}_g^+(u) < M$. Suppose first that $|I_i| \leq 1/\sqrt{\beta}$ and $0 \leq i \leq p-1$. We then compute

$$\|u'\|^2_{L^2(I_i)} \leq |I_i|^2 \|u''\|^2_{L^2(I_i)} \leq \frac{1}{\beta}\|u''\|^2_{L^2(I_i)}.$$

It follows that

$$\mathcal{F}_g|_{I_i}(u) \geq \frac{1}{2}(\|u''\|^2_{L^2(I_i)} - \beta\|u'\|^2_{L^2(I_i)}) \geq 0.$$

Suppose now that $|I_i| > 1/\sqrt{\beta}$ (which is of course the case for I_p) and assume to fix the ideas that u is positive in the interval I_i. The case of an interval where u is negative follows by symmetry. We deduce from Lemma 3.11 that

$$\begin{aligned}\mathcal{F}_g|_{I_i}(u) &\geq \int_{x_i}^{x_{i+1}} \left(\frac{1}{2}(u''^2 - \beta u'^2) + k(u-1)^2\right) dx \\ &\geq \varepsilon\|u-1\|^2_{H^2(I_i)} - 2(\varepsilon + \frac{\beta}{2})\|u'\|_{L^\infty(I_i)}. \qquad (4.10)\end{aligned}$$

Indeed, we have $u(x_i) = u(x_{i+1}) = 0$ for $i = 0, \ldots, p-1$ while remember that $x_{p+1} = +\infty$ and $u(x_{p+1}) = 1$.

Combining the estimate (4.10) with the inequality (3.15) of Lemma 3.12 applied to u' with $b - a \geq 1/\sqrt{\beta}$, we obtain

$$\mathcal{F}_g|_{I_i}(u) \geq \varepsilon\|u-1\|^2_{H^2(I_i)} - 2C(1+\sqrt{\beta})(\varepsilon + \frac{\beta}{2})\|u-1\|_{H^2(I_i)} \quad (4.11)$$

so that the conclusion easily follows. $\square$

From Lemma 4.6, we now deduce a lower estimate on the length of the intervals I_i.

Lemma 4.7. *Let $M > 0$ and assume that f and $g \in C(\mathbb{R})$ satisfy assumption* (C2). *There exists $\eta > 0$ such that if $u \in \mathcal{E}_p^+$ satisfies $\mathcal{F}_g^+(u) < M$, then $|I_i| \geq \eta$ for all $i = 0, \ldots, p$.*

Proof. Let $u \in \mathcal{E}_p^+$ be such that $\mathcal{F}_g^+(u) < M$. Let

$$\eta = \min(\sqrt[3]{\frac{3}{2(M+pK)}}, \frac{1}{\sqrt{2\beta}}),$$

where K is given by Lemma 4.6 and suppose by contradiction that $|I_{i_0}| < \eta$ for some $0 \leq i_0 \leq p-1$. As $\eta \leq \frac{1}{\sqrt{2\beta}}$, we infer that

$$\|u'\|^2_{L^2(I_{i_0})} \leq \frac{1}{2\beta}\|u''\|^2_{L^2(I_{i_0})}$$

and

$$\mathcal{F}_g|_{I_{i_0}}(u) \geq \frac{1}{4}\|u''\|^2_{L^2(I_{i_0})}. \quad (4.12)$$

We now estimate the L^2-norm of u'' by means of the inequality of Lemma 2.5. To fix the ideas, we suppose again that u is positive on I_{i_0}. By definition of $\mathcal{E}_p^+$, we know that for some $\bar{x}_{i_0} \in I_{i_0}$, $u(\bar{x}_{i_0}) = \max_{I_{i_0}} u > 1$. Taking $A = u(x_{i_0}) = 0$, $B = u(\bar{x}_{i_0})$, $A_1 = u'(x_{i_0})$, $B_1 = u'(\bar{x}_{i_0}) = 0$ and $\tau_{i_0} = \bar{x}_{i_0} - x_{i_0}$, we deduce from Lemma 2.5 that

$$\int_{x_{i_0}}^{\bar{x}_{i_0}} u''^2\, dx \geq \frac{4}{\tau_{i_0}}\Big((u'(x_{i_0}))^2 + 3(\frac{u(\bar{x}_{i_0})}{\tau_{i_0}} - u'(x_{i_0}))\frac{u(\bar{x}_{i_0})}{\tau_{i_0}}\Big) \geq \frac{3u^2(\bar{x}_{i_0})}{\tau_{i_0}^3}.$$

A similar inequality holds for the integral between $\bar{x}_{i_0}$ and x_{i_0+1} so that, taking (4.12) into account, we obtain

$$\mathcal{F}_g|_{I_{i_0}}(u) \geq \frac{3u^2(\bar{x}_{i_0})}{2|I_{i_0}|^3}.$$

Now, as $\mathcal{F}_g|_{I_i}(u) \geq -K$ for all other I_i's, we come out with a contradiction as

$$\mathcal{F}_g^+(u) \geq \frac{3}{2\eta^3} - pK \geq M.$$

□

The previous lower estimate on the length of the intervals I_i implies, for every function $u \in \mathcal{E}_p^+$ that satisfies $\mathcal{F}_g^+(u) < M$, a uniform upper bound on either $\|u-1\|_{H^2(I_i)}$ or $\|u+1\|_{H^2(I_i)}$.

Lemma 4.8. *Let $M > 0$ and assume f and $g \in C(\mathbb{R})$ satisfy assumption* (C2). *Then, there exists $N > 0$ such that if $u \in \mathcal{E}_p^+$ satisfies $\mathcal{F}_g^+(u) < M$, for all $i = 0, \ldots, p$,*

$$\|u-1\|_{H^2(I_i)} \leq N$$

on every interval I_i where u is positive, while

$$\|u+1\|_{H^2(I_i)} \leq N$$

for an interval I_i where u is negative.

Proof. Let $u \in \mathcal{E}_p^+$ be such that $\mathcal{F}_g^+(u) < M$. We first deduce from Lemma 4.7 the existence of $\eta > 0$ such that $|I_i| \geq \eta$ for all $i = 0, \ldots, p$. We can therefore obtain an estimate similar to (4.11) when u is positive in I_i. An estimate of the H^2-norm of $u+1$ holds in intervals I_i where u is negative.

From Lemma 4.6, we also know that for some positive constant K, $\mathcal{F}_g|_{I_i}(u) \geq -K$ for all $i = 0, \ldots, p$. It follows that for each $i = 0, \ldots, p$, we have

$$\mathcal{F}_g|_{I_i}(u) \leq \mathcal{F}_g^+(u) + pK.$$

Therefore, if u is positive in I_i, we deduce, for some constant $C > 0$, the estimate

$$\varepsilon\|u-1\|^2_{H^2(I_i)} - C\|u-1\|_{H^2(I_i)} \leq M + pK$$

and similarly

$$\varepsilon\|u+1\|^2_{H^2(I_i)} - C\|u+1\|_{H^2(I_i)} \leq M + pK$$

on an interval I_i where u is negative. These two estimates imply the desired a priori bounds. □

The success of the minimization process depends on a control on the length of the intervals I_i to avoid a loss of compactness. In fact, even for a function $u \in \mathcal{E}_p^+$ that satisfies $\mathcal{F}_g^+(u) < M$, we cannot obtain a bound on the length of the intervals I_i^u because if u is very close to ± 1 with small derivatives on a long interval, the action along this interval can be arbitrarily small. However, we obtain a control on the length of the transitions of any function of a minimizing sequence by locally modifying it where it is too close to one of the equilibria. Such ideas were developed in the second part of the proof of Theorem 3.10. The key arguments for such modifications are the oscillatory nature of the minimizers close to the equilibria and the clipping procedure.

LEMMA 4.9. *Let f, $g \in C^2(\mathbb{R})$ be even functions such that $f(1) = 0$, $g^2(1) < 4f''(1)$ and assume that* (C2) *holds. Let $M > 0$ and $u \in \mathcal{E}_p^+$ be such that $\mathcal{F}_g^+(u) < M$. Then there exists $I > 0$ and $v \in \mathcal{E}_p^+$ such that for all $i = 0, \dots, p-1$, $|I_i^v| \le I$ and $\mathcal{F}_g^+(v) \le \mathcal{F}_g^+(u)$.*

The proof of this lemma follows the lines of Step 2 in Part 2 of the proof of Theorem 3.10. The needed a priori bounds can be deduced from the preceding lemmas.

As a direct consequence of Lemma 4.9, we deduce that the right endpoint of the interval I_{p-1} does not go to infinity along a minimizing sequence. This fact is of course essential for the construction of a minimizing sequence that satisfies properties (a) and (b). The following proposition summarizes in some sense all the previous lemmas.

PROPOSITION 4.10. *Assume f and $g \in C^2(\mathbb{R})$ are even functions such that $f(1) = 0$, $g^2(1) < 4f''(1)$ and* (C2) *holds. Then, for each $p \in \mathbb{N}$, there exists $I > 0$, $C > 0$ and a minimizing sequence $(u_n)_n \subset \mathcal{E}_p^+$ such that for all $n \in \mathbb{N}$, for all $i = 0, \dots, p-1$,*

$$|I_i^{u_n}| \le I \tag{4.13}$$

and

$$\|u_n - 1\|_{H^2(\mathbb{R}^+)} \le C. \tag{4.14}$$

PROOF. Let $p \in \mathbb{N}$ be fixed. From Lemma 4.6, we know that

$$c = \inf_{\mathcal{E}_p^+} \mathcal{F}_g^+ \ge -(p+1)K$$

for some positive constant K. Let $(u_n)_n \subset \mathcal{E}_p^+$ be a minimizing sequence. We may assume without loss of generality that for all $n \in \mathbb{N}$,

$$\mathcal{F}_g^+(u_n) < c + 1.$$

Therefore, all the previous lemmas, with $M = c + 1$, apply to the functions u_n. It follows from Lemma 4.9 that we may choose the sequence $(u_n)_n \subset \mathcal{E}_p^+$ in such a way that (4.13) holds.

To prove estimate (4.14), we first observe that x_p, the right endpoint of I_{p-1}, is such that $x_p \leq pI$. On the other hand, we also know from Lemma 4.7 and Lemma 4.8 that there exist $\eta > 0$ and $N > 0$ such that $|I_i^{u_n}| \geq \eta$ for all $i = 0, \dots, p$ and all $n \in \mathbb{N}$, and

$$\|u_n - 1\|_{H^2(I_i^{u_n})} \leq N$$

on any interval $I_i^{u_n}$ where u_n is positive, while

$$\|u_n + 1\|_{H^2(I_i^{u_n})} \leq N$$

if u_n is negative in the interval $I_i^{u_n}$. Now, by means of inequality (3.15) and the fact that $|I_i^{u_n}| \geq \eta$, we deduce the existence of a positive constant R such that

$$\|u_n\|_{L^\infty(\mathbb{R}^+)} \leq R. \tag{4.15}$$

Denoting by J^+ (respectively J^-) the set of indexes $i \in (0, \dots, p)$ corresponding to intervals $I_i^{u_n}$ where u_n is positive (respectively negative), we have

$$\begin{aligned}
\|u_n - 1\|^2_{H^2(\mathbb{R}^+)} &= \int_{\mathbb{R}^+} \left(u_n''^2 + u_n'^2 + (u_n - 1)^2\right) dx \\
&= \sum_{i \in J^+} \int_{I_i^{u_n}} \left(u_n''^2 + u_n'^2 + (u_n - 1)^2\right) dx \\
&\quad + \sum_{i \in J^-} \int_{I_i^{u_n}} \left(u_n''^2 + u_n'^2 + (u_n + 1)^2 - 4u_n\right) dx \\
&\leq (p+1)N - 4 \sum_{i \in J^-} \int_{I_i^{u_n}} u_n.
\end{aligned}$$

Taking the estimates (4.13) and (4.15) into account, we finally conclude that

$$\|u_n - 1\|^2_{H^2(\mathbb{R}^+)} \leq (p+1)N + 4mRI$$

where m is the number of indexes in J^-. □

Existence of the local minimizers. We have now almost all the ingredients to prove Theorem 4.4. To complete the minimizing process in a given class $\mathcal{E}_p^+$, we choose a minimizing sequence $(u_n)_n \subset \mathcal{E}_p^+$ that has the properties described in Proposition 4.10. From the a priori bound on $\|u_n - 1\|_{H^2(\mathbb{R}^+)}$, we deduce that up to a subsequence, u_n converges (in a way which is made precise below) to some function $u \in \mathcal{E}^+$. To prove that $u \in \mathcal{E}_p^+$, we require $\mathcal{F}_g^+$ to be lower bounded in $\mathcal{E}^+$. Observe that this last assumption was not used until now and actually, we do not know whether it is necessary or not.

The idea of the next lemma is contained in the proof of Theorem 3.6. We shortly recall it.

LEMMA 4.11. *Assume that f and $g \in C(\mathbb{R})$ satisfy assumption* (C2). *Let $u \in H^2(a,b)$ be such that $u(a) = u(b) = 0$. Assume moreover that $\inf_{\mathcal{E}^+} \mathcal{F}_g^+ > -\infty$. Then*

$$\mathcal{F}_g|_a^b(u) = \int_a^b \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \geq 0.$$

PROOF. Let $v \in \mathcal{E}^+$ be such that $v'(0) = u'(b)$. Assume by contradiction that $\mathcal{F}_g|_a^b(u) = -c < 0$ and let $p \in \mathbb{N}$ be such that

$$\mathcal{F}_g^+(v) - (2p+1)c < \inf_{\mathcal{E}^+} \mathcal{F}_g^+.$$

Then, we set $T = b - a$ and define the function $\bar{u} \in \mathcal{E}^+$ by

$$\bar{u}(x) = \begin{cases} u(a+x) & \text{if } x \in [0,T), \\ -u(a+2iT-x) & \text{if } x \in [(2i-1)T, 2iT),\ i = 1, \cdots, p, \\ u(a+x-2iT) & \text{if } x \in [2iT, (2i+1)T),\ i = 1, \cdots, p, \\ v(x-(2p+1)T) & \text{if } x \geq (2p+1)T. \end{cases}$$

We now derive a contradiction as

$$\mathcal{F}_g^+(\bar{u}) = (2p+1)\mathcal{F}_g|_a^b(u) + \mathcal{F}_g^+(v) < \inf_{\mathcal{E}^+} \mathcal{F}_g^+.$$

□

PROOF OF THEOREM 4.4. Let $(u_n)_n \subset \mathcal{E}_p^+$ be a minimizing sequence that has the properties stated in Proposition 4.10. Let $\delta_0 > 0$ and $\tau_0 > 0$ be given by Lemma 3.14.

Step 1 - The sequence $(u_n)_n$ converges in C^1_{loc} to some function $u \in \mathcal{E}^+$ that satisfies

$$\mathcal{F}_g^+(u) \leq \inf_{\mathcal{E}_p^+} \mathcal{F}_g^+. \tag{4.16}$$

From the uniform bound on $\|u_n - 1\|_{H^2(\mathbb{R}^+)}$, we deduce that up to a subsequence,

$$u_n - 1 \overset{H^2(\mathbb{R}^+)}{\rightharpoonup} u - 1 \quad \text{and} \quad u_n \xrightarrow{C^1_{\text{loc}}} u$$

for some function $u \in \mathcal{E}^+$. We now proceed as in the proof of Theorem 3.10 to deduce inequality (4.16). Let $J_1 = [0, pI]$ and $J_2 = (pI, +\infty)$ where I is given by Proposition 4.10. We then write

$$\begin{aligned} \mathcal{F}_g^+(u_n) &= \mathcal{F}_g|_{J_1}(u_n) + \mathcal{F}_g|_{J_2}(u_n) \\ &= \int_{J_1} \left(\frac{1}{2}(u_n''^2 + g(u_n){u_n'}^2) + f(u_n) \right) dx \\ &\quad + \int_{J_2} \left(\frac{1}{2}(u_n''^2 + g(u_n){u_n'}^2) + f(u_n) \right) dx. \end{aligned}$$

Observe first that

$$\mathcal{F}_g|_{J_1}(u) \leq \liminf_{n\to+\infty} \mathcal{F}_g|_{J_1}(u_n). \tag{4.17}$$

On J_2, we know that u_n is positive. We therefore write

$$\begin{aligned}
\mathcal{F}_g|_{J_2}(u_n) &= \tfrac{1}{2}\int_{J_2} \left(u_n''^2 - \beta u_n'^2 + \frac{\beta^2}{4}(u_n-1)^2 \right) dx \\
&\quad + \int_{J_2} \frac{g(u_n)+\beta}{2}\, u_n'^2\, dx + \int_{J_2} \left(f(u_n) - \frac{\beta^2}{8}(u_n-1)^2 \right) dx \\
&= \frac{1}{2}\left(\int_{J_2} \left(u_n'' + \frac{\beta}{2}(u_n-1) \right)^2 dx - \beta u_n'(pI)u_n(pI) \right) \\
&\quad + \int_{J_2} \frac{g(u_n)+\beta}{2} u_n'^2\, dx + \int_{J_2} \left(f(u_n) - \frac{\beta^2}{8}(u_n-1)^2 \right) dx.
\end{aligned}$$

Observe that in the last equality, the first integral is convex. Since Fatou's Lemma is applicable to the last two, we conclude, using inequality (4.17), that

$$\mathcal{F}_g^+(u) \leq \inf_{\mathcal{E}_p^+} \mathcal{F}_g^+.$$

We denote the extremities of the intervals $I_i^{u_n}$ by x_i^n, $i = 0, \ldots, p$. It is clear by uniform convergence that up to a subsequence, x_i^n converges to some $x_i \in J_1$, for all $i = 1, \ldots, p$. Remember that by convention, we set $x_0^n = x_0 = 0$ and $x_{p+1}^n = x_{p+1} = +\infty$. We denote by I_i the intervals $]x_i, x_{i+1}[$, $i = 0, \ldots, p$. We also deduce from the convergence in $C^1(J_1)$ and the convergence in $C^1_{\text{loc}}(J_2)$ that

$$u(x)(-1)^{i+p} \geq 0 \text{ for } x \in I_i$$

and

$$\max_{x\in I_i} u(x)(-1)^{i+p} \geq 1.$$

Step 2 - Elimination of the zeros of u after x_p. If u has zeros after x_p, we first modify it to keep only one of these zeros. So, suppose that u vanishes at least two times after x_p. We then define

$$a_1 = \min\{x > x_p \mid u(x) = 0\} \text{ and } a_2 = \max\{x > x_p \mid u(x) = 0\}.$$

Observe that a_2 is well defined because $u \in \mathcal{E}^+$. By convergence in C^1_{loc}, $u'(a_1) = u'(a_2) = 0$ so that the interval $[a_1, a_2]$ can be clipped out and the resulting function has only one zero after x_p. Moreover the function u is non-negative on the clipped interval so that this modification decreases the action.

Assume now that u vanishes at some point $\xi > x_p$. We then have $u'(\xi) = 0$. Now as $u(x_p) = u(\xi)$, there exists at least one critical point y between x_p and ξ such that $u(y) > 0$. Here, we have two possibilities, either y can be taken in such a way that $u(y) \leq 1$ or $[0,1]$ does not contain any critical value of $u_{|[x_p,\xi]}$.

Case 1. Suppose first that we can find $y \in (x_p, \xi)$ such that $0 < u(y) \leq 1$ and $u'(y) = 0$. As $u \in \mathcal{E}^+$, we know that

$$\lim_{x \to +\infty} (u(x), u'(x)) = (1, 0).$$

Let T be such that $(u(x), u'(x)) \in B((1,0), \delta_0)$ for all $x \geq T$. Then, the restriction of u to the interval $[T, +\infty[$ clearly minimizes the functional

$$\int_T^{+\infty} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx$$

in the set of functions $\varphi \in H^2_{loc}([T, +\infty[)$ satisfying $\varphi - 1 \in H^2(T, +\infty)$ and $(\varphi(T), \varphi'(T)) = (u(T), u'(T))$. We therefore infer from Lemma 3.14 that $u(x)$ oscillates around 1 for x large enough. Consequently, we are able to find a local maxima of u above $+1$ and clip out an interval containing $[y, \xi]$ in such a way that the corresponding clip of u does not vanish after x_p.

Case 2. In the second case, we can find $y \in (x_p, \xi)$ such that $u(y) > 1$ and if $x \in (x_p, \xi)$ satisfies $u'(x) = 0$, then $u(x) > 1$. We now define $v \in \mathcal{E}^+$ by

$$v(x) = \begin{cases} -u(x + x_1) & \text{if } 0 \leq x \leq \xi - x_1 \\ u(x + x_1) & \text{if } x > \xi - x_1. \end{cases}$$

Observe that since $\min_{[x_p - x_1, \xi - x_1]} v(x) < -1$ and v is negative in the interval $(x_p - x_1, \xi - x_1)$, v has the right number of transitions. Also, v does not vanish after $\xi - x_1$. On the other hand, since $u(x_0) = u(x_1)$, we deduce from Lemma 4.11 that

$$\int_{x_0}^{x_1} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx \geq 0$$

so that $\mathcal{F}_g^+(v) \leq \mathcal{F}_g^+(u)$.

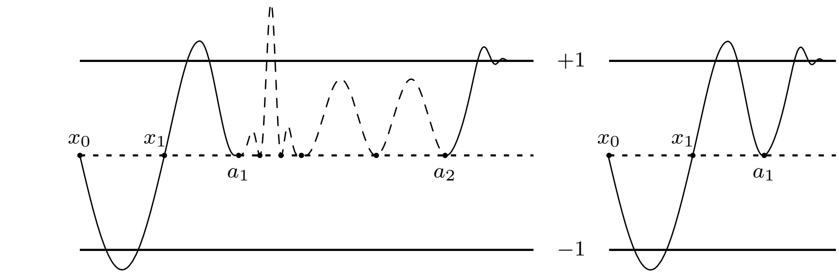
Step 3 - Elimination of the zeros of v in the bumps. We still denote by $0 = x_0 < x_1 < \ldots < x_p$, the extremities of the intervals I_i^v (actually, these are the intervals I_i^u which have been possibly translated in Step 2). Suppose that there exists $\xi \in v^{-1}(0)$ so that $\xi \neq x_i$ for any $i = 0, \ldots, p$. Hence, ξ lies in an interval I_i. To fix the ideas, we assume that v is non-negative therein and denoting by $\bar{x}_i$ the maximum of v over this interval we assume that ξ is at the left of $\bar{x}_i$. Next, define

$$\xi_1 = \min\{x \in I_i \mid v'(x) = 0\}$$

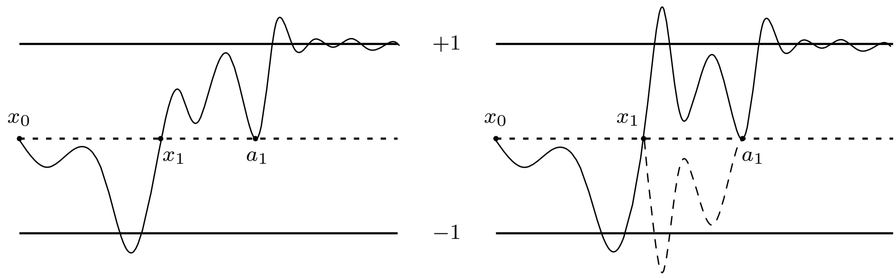
and

$$\xi_2 = \max\{x \in [\xi_1, \bar{x}_i] \mid v(x) = 0\}.$$

It is easily seen that an interval containing $[\xi_1, \xi_2]$ can be clipped out so that the zeros can be removed.



Elimination of the zeros between a_1 and a_2. The piece of graph between a_1 and a_2 may be thrown away and the remaining parts glued together.



Elimination of a_1. In Case 1, on the left-hand side, we may clearly use the clipping process to discard an interval containing a_1. In Case 2, on the right-hand side, we drop the first negative bump, we flip the graph between x_1 and a_1 over and then translate x_1 to 0*.*

FIGURE 4.1. *Step* 2 *of the proof of Theorem* 4.4*.*

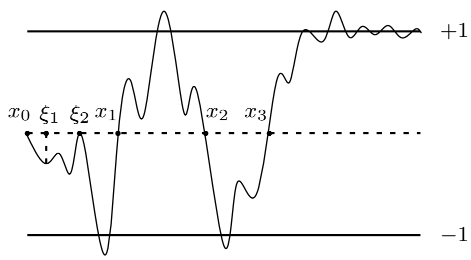


FIGURE 4.2. *Step* 3 *- Elimination of the zeros in the bumps. An interval containing $[\xi_1, \xi_2]$ may obviously be clipped out.*

Step 4 - Elimination of the tangencies with ± 1. The last condition that we have to check to be sure that $v \in \mathcal{E}_p^+$ is that

$$\max_{x \in I_i} |v(x)| > 1$$

for all $i = 0, \ldots, p$. Assume that this condition fails to be true in one of the intervals I_i. In this interval, we thus have

$$\max_{x \in I_i} |v(x)| = 1.$$

Let $\tau \in I_i$ be such that $|v(\tau)| = 1$ and $v'(\tau) = 0$. To fix the ideas, assume that $v(\tau) = 1$. The second case may be handled in the same way. As the action of the function 1 is zero, we can modify v without increasing its action by stretching the point τ to an interval of arbitrary length and gluing the function 1 to both extremities (this idea was first used in W. D. Kalies et al. [**54**]). Now, we take a_1 and a_2, respectively at the left and at the right of τ in order to have

$$0 < \max_{i=1,2} \|(v(a_i) - 1, v'(a_i))\| \leq \delta_0$$

and we stretch τ to an interval of length τ_0. We still call v the function obtained after gluing 1 at τ and $\tau + \tau_0$. It follows from Lemma 3.14 that the minimizers of the functional

$$\int_{a_1}^{a_2+\tau_0} \left(\frac{1}{2}(u''^2 + g(u)u'^2) + f(u) \right) dx$$

defined on the set of functions $\varphi \in H^2(a_1, \tau_0 + a_2)$ that satisfy the boundary conditions $(\varphi(a_1), \varphi'(a_1)) = (v(a_1), v'(a_1))$ and $(\varphi(a_2 + \tau_0), \varphi'(a_2 + \tau_0)) = (v(a_2), v'(a_2))$, do oscillate around 1. Hence, if we locally replace v between a_1 and $a_2 + \tau_0$ by a minimizer, we obtain a new function w such that

$$\max_{x \in I_i} |w(x)| > 1$$

and $\mathcal{F}_g^+(w) \leq \mathcal{F}_g^+(v)$.

Conclusion - It follows from the previous steps that there exists $w \in \mathcal{E}_p^+$ such that $\mathcal{F}_g^+(w) \leq \mathcal{F}_g^+(u)$. Consequently, we have $\mathcal{F}_g^+(w) = \min_{\mathcal{E}_p^+} \mathcal{F}$. Now, observe that for all $h \in H^2(\mathbb{R}^+)$ such that $h(0) = 0$, for t sufficiently small, $\mathcal{F}_g^+(w) \leq \mathcal{F}_g^+(w + th)$. Indeed, assume that there exists a sequence $(t_n)_n$ tending to 0 such that $\mathcal{F}_g^+(w) > \mathcal{F}_g^+(w + t_n h)$. If w is in the interior of the class $\mathcal{E}_p^+$, this is obviously a contradiction. In the case where w is on the boundary of $\mathcal{E}_p^+$, i.e. if for some points x_i, $w(x_i) = w'(x_i) = 0$, then even for t_n small, $w + t_n h$ may have more than one zero close to the x_i's so that it does not necessarily belong to $\mathcal{E}_p^+$. However for t_n small enough, $w + t_n h$ has the right number of transitions and we are able to remove the oscillations close to the points x_i by means of the clipping procedure. Therefore, for n large enough,

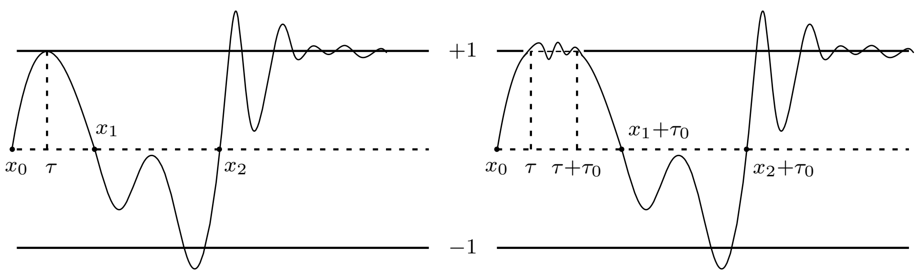


FIGURE 4.3. *Step 4 - Elimination of the tangencies with ± 1. We first stretch the point τ to an interval of length τ_0 and glue the function 1 at the extremities of the interval $[\tau, \tau+\tau_0]$. Then we locally modify the function on an interval of length slightly larger than τ_0.*

modifying $w + t_n h$ close to the x_i's if necessary, we obtain a function in $\mathcal{E}_p^+$ whose action is strictly smaller than $\mathcal{F}_g^+(w)$. This contradicts the definition of w.

We now deduce that w is a critical point of $\mathcal{F}_g^+$. Using by now familiar arguments, it is easily shown that w satisfies equation (4.6) in $\mathbb{R}^+$ and we finally conclude that the odd extension w^* of w is a heteroclinic solution of (4.6). Notice that, using the conservation of the Hamiltonian

$$H(u) = u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u)$$

along solutions of (4.6), it is easy to check that each local minimizer is actually in the interior of $\mathcal{E}_p^+$, i.e. each crossing with zero is transverse. □

As already asserted, the hypothesis $\inf_{\mathcal{E}^+} \mathcal{F}_g^+ > -\infty$ of Theorem 4.4 implies that $\inf_{\mathcal{E}^+} \mathcal{F}_g^+ \geq 0$, see Theorem 3.7. Actually, we can even prove that $\inf_{\mathcal{E}^+} \mathcal{F}_g^+ > 0$ and consequently the multi-transition heteroclinics all have a strictly positive action. Observe also that the analysis of Section 2.3 applies to the framework we are considering by now. Thus, the local minimizer u_p of $\mathcal{F}_g^+$ in $\mathcal{E}_p^+$ has exactly one critical point in each interval I_i for every $i = 0, \ldots, p-1$. On the other hand, in the tail, u_p oscillates around 1.

Let us come back to the stationary Swift-Hohenberg equation (4.1). In Theorem 3.9, we show that $\inf_{\mathcal{E}} \mathcal{F}_\beta = -\infty$ for $\beta < \beta_0$ where $\mathcal{E}$ is defined by (4.3) and

$$\beta_0 = \inf\{\beta < 0 | \inf_{\mathcal{E}} \mathcal{F}_\beta \geq 0\}. \tag{4.18}$$

For $\beta \geq \beta_0$, we obtain a family of odd heteroclinics u_p^* having $2p+1$ zeros in $\mathbb{R}$. We still denote by $\mathcal{F}_\beta^+ : \mathcal{E}^+ \to \mathbb{R}$ the functional defined by

$$\mathcal{F}_\beta^+(u) = \int_{\mathbb{R}^+} \left(\frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2-1)^2 \right) dx. \tag{4.19}$$

PROPOSITION 4.12. *Let β_0 be defined by* (4.18)*. For all $\beta \geq \beta_0$, for all $p \in \mathbb{N}$, the functional $\mathcal{F}_\beta^+ : \mathcal{E}^+ \to \mathbb{R}$ defined by* (4.19) *and* (4.8)*, has a local minimum $u_p \in \mathcal{E}_p^+$ whose odd extension is a solution of* (4.1) *connecting -1 to $+1$ and having exactly $2p+1$ zeros. Moreover, $\mathcal{F}_\beta^+(u_p) < \mathcal{F}_\beta^+(u_{p+1})$ for all $p \in \mathbb{N}$.*

PROOF. The existence of the minimizers $u_p \in \mathcal{E}_p^+$ follows directly from Theorem 4.4 and the definition of β_0. For all $p \in \mathbb{N}$, we denote by u_p^* the odd extension of u_p in $\mathbb{R}$. We now prove that for all $p \in \mathbb{N}$, $\mathcal{F}_\beta^+(u_p) < \mathcal{F}_\beta^+(u_{p+1})$. Let x_1 be the first zero of u_{p+1} in $\mathbb{R}^+$. Then, the function $v_p \in \mathcal{E}^+$ defined by $v_p(x) = u_{p+1}(x + x_1)$ belongs to $\mathcal{E}_p^+$. It follows from Lemma 4.11 that the action of u_{p+1} on the interval $[x_0, x_1]$ is non-negative. Therefore $\mathcal{F}_\beta^+(u_p) \leq \mathcal{F}_\beta^+(v_p) \leq \mathcal{F}_\beta^+(u_{p+1})$. Suppose that $\mathcal{F}_\beta^+(u_p) = \mathcal{F}_\beta^+(u_{p+1})$. Then v_p is a minimizer of $\mathcal{F}_\beta^+$ in $\mathcal{E}_p^+$ and thus its odd extension v_p^* is a heteroclinic solution of (4.1). Observe that $u_{p+1}^*(\cdot + x_1)$ also solves (4.1) in $\mathbb{R}$. As $u_{p+1}^*(\cdot + x_1) = v_p^*$ in $\mathbb{R}^+$, this contradicts the uniqueness of the solution of the Cauchy problem. $\square$

4.2.2. Functionals with Sign Changing Acceleration Coefficient. In this section, we consider the setting of Section 3.1 assuming in addition that the functional is symmetric and the equilibria are saddle-foci.

THEOREM 4.13. *Let $f \in C^2(\mathbb{R})$ be a non-negative even function that satisfies assumptions* (B1) *and* (B3)*. Assume $g \in C^2(\mathbb{R})$ is an even function that satisfies $g(1)^2 < 4f''(1)$ and assumption* (C1)*. Then, the conclusion of Theorem* 4.4 *holds.*

The proof of Theorem 4.13 require an adaption of Lemma 3.14. We recall the definition of the set

$$\mathcal{E}_{[a,b]}(\eta) = \{u \in \mathcal{E}_{[a,b]} \mid \|u\|_{L^\infty(a,b)} \leq \eta\},$$

where

$$\mathcal{E}_{[a,b]} = \{u \in H^2(a,b) \mid (u(a), u'(a)) = y_0 \text{ and } (u(b), u'(b)) = y_1\}.$$

LEMMA 4.14. *Let f and $g \in C^2(\mathbb{R})$ be such that $f(u) \geq 0$ for all $u \in \mathbb{R}$, $f(0) = f'(0) = 0$ and assume* (C1) *holds. Assume moreover that $f''(0) > 0$. Then, there exist $\eta > 0$, $\delta_0 > 0$ and $S > 1$ such that if*

$b-a \geq 1$ and $\max(\|y_0\|, \|y_1\|) \leq \delta_0$, the functional $\mathcal{F}_g|_a^b$ has a minimizer $u \in \mathcal{E}_{[a,b]}(\eta)$ such that

$$\|u\|_{C^3([a,b])} \leq S \max(\|y_0\|, \|y_1\|).$$

Moreover, if $g(0)^2 < 4f''(0)$, then there exists τ_0 such that if $b-a \geq 1$, u changes sign on every subinterval of $[a,b]$ having length larger than τ_0.

PROOF. As 0 is a nondegenerate minimum of f, there exist $\eta > 0$, $\theta > 0$ and $\zeta > 0$ such that $|f'(u)| \leq 2\theta|u|$ and $\zeta u^2 \leq f(u) \leq \theta u^2$ for $|u| \leq \eta$. Notice also that using integration by parts and arguing as in Lemma 3.1, we see that there exists $s > 0$ such that for any function $u \in H^2(a,b)$,

$$\mathcal{F}_g|_a^b(u) \geq s \int_a^b \left(\frac{u''^2}{2} + f(u) \right) dx + \frac{1}{2}(\tilde{G}(u(b))u'(b) - \tilde{G}(u(a))u'(a)), \tag{4.20}$$

so that $\mathcal{F}_g|_a^b$ is bounded from below in $\mathcal{E}_{[a,b]}(\eta)$. Hence, the existence of a minimizer follows by standard arguments.

Claim 1 - There exists $C_1 > 0$ and $\delta_1 > 0$ such that if $\|y_0\| \leq \delta \leq \delta_1$ and $\|y_1\| \leq \delta \leq \delta_1$, then $\inf_{\mathcal{E}_{[a,b]}(\eta)} \mathcal{F}_g|_a^b \leq C_1\delta^2$. The proof of this claim is similar to that of Step 1 in the proof of Lemma 3.13.

Claim 2 - There exists $\delta_2 > 0$ such that if u is a minimizer in $\mathcal{E}_{[a,b]}(\eta)$ with $\|y_0\| \leq \delta_2$ and $\|y_1\| \leq \delta_2$, then $\|u\|_\infty \leq \eta/2$. The ideas we use to prove the claim are already included in the proof of Proposition 2.9. Observe that the minimizers of $\mathcal{F}_g|_a^b$ in $\mathcal{E}_{[a,b]}(\eta)$ are a priori bounded in $C^1([a,b])$. Indeed, this follows easily arguing as in the first part of the proof of Proposition 2.9. We denote by L a C^1-bound. Consider next the set $\mathcal{E}_{[a,b]}(\eta/2)$. Let us fix the notation $N_{\eta/2} := [-\eta/2, \eta/2]$. We then define

$$\mu := \min\{f(u) \mid u \in N_{\eta/2} \setminus N_{\frac{\eta}{4}}\} > 0,$$
$$\nu := \max\{|\tilde{G}(u)| \mid u \in N_{\eta/2}\} \geq 0.$$

As in Step 2 of Part 2 in the proof of Proposition 2.9, we can derive a lower estimate on the action of functions $u \in \mathcal{E}_{[a,b]}(\eta)$ whose graphs do not stay in the strip $[a,b] \times N_{\eta/2}$. Indeed, suppose $u \notin \mathcal{E}_{[a,b]}(\eta/2)$ minimizes $\mathcal{F}_g|_a^b$ in $\mathcal{E}_{[a,b]}(\eta)$ with $\|y_0\| \leq \delta \leq \delta_1$ and $\|y_1\| \leq \delta \leq \delta_1$. Then

$$\mathcal{F}_g|_a^b(u) \geq \frac{s\mu\eta}{4L} - \nu\delta. \tag{4.21}$$

On the other hand, we infer from Claim 1 that

$$\mathcal{F}_g|_a^b(u) \leq C_1\delta^2.$$

Choosing $\delta_2 > 0$ small enough and $0 < \delta \leq \delta_2$, the estimate (4.21) yields a contradiction so that $u \in \mathcal{E}_{[a,b]}(\eta/2)$.

It follows from Claim 2 that if δ_0 is sufficiently small and both $\|y_0\|$ and $\|y_1\|$ are smaller that δ_0, the minimizer u of $\mathcal{F}_g|_a^b$ in $\mathcal{E}_{[a,b]}(\eta)$ solves (4.6) in $[a,b]$ together with the boundary conditions $(u(a),u'(a)) = y_0$ and $(u(b),u'(b)) = y_1$. The remaining of the proof is identical to these of Lemma 3.13 and Lemma 3.14. $\square$

We now turn to the proof of Theorem 4.13. Since many of the arguments are by now familiar, we only sketch it.

Proof of Theorem 4.13. First, observe that assumption (C1) implies that $\mathcal{F}_g^+(u) \geq 0$ for all $u \in \mathcal{E}^+$. This follows from the inequality

$$\mathcal{F}_g^+(u) \geq s \int_{\mathbb{R}^+} \left(\frac{u''^2}{2} + f(u) \right) dx,$$

where $s > 0$, which is valid for all $u \in \mathcal{E}^+$ and proved in Lemma 3.1.

Let $(u_n)_n \subset \mathcal{E}_p^+$ be a minimizing sequence for $\mathcal{F}_g^+$. For $i = 1, \ldots, p$, the inequality

$$\mathcal{F}_g|_{I_i}(u_n) \geq s \int_{I_i} \left(\frac{u_n''^2}{2} + f(u_n) \right) dx \tag{4.22}$$

follows arguing as in Lemma 3.1. Consequently, the equivalent of Lemma 4.6 (with $K = 0$) and Lemma 4.7 hold. Next, we deduce, as in Proposition 2.9, that $\|u_n\|_\infty$ and $\|u_n'\|_\infty$ are uniformly bounded. Then, using the inequality (4.22), these a priori bounds and the clipping procedure, the conclusion of Lemma 4.9 follows (possibly for a modification of the minimizing sequence). Therefore, the statement of Lemma 4.8 for $i = 0, \ldots, p-1$ holds. To obtain an a priori bound on the length of I_p, we first observe that for some $a > 0$, $k > 0$ and $\beta \in [0, \sqrt{8k})$,

$$f(u) \geq k(u-1)^2 \ \text{ and } \ g(u) \geq -\beta \quad \text{for } |u-1| \leq a. \tag{4.23}$$

Then, arguing as in Proposition 2.9 and using the bound on I_i for every $i = 0, \ldots, p-1$, we may assume that there exists $T > 0$ such that for every $n \in \mathbb{N}$ and all $x \geq T$, u_n satisfies

$$|u_n(x) - 1| \leq a.$$

We now deduce an a priori bound for $\|u_n - 1\|_{H^2(x_p,+\infty)}$ using the uniform bounds on u_n and u_n' in $[x_p, T]$ and the estimate of Lemma 3.11 in the interval $]T, +\infty[$.

We thus obtain a minimizing sequence $(u_n)_n \subset \mathcal{E}_p^+$ that has the properties of Proposition 4.10. Now, the end of the proof follows the same lines as the proof of Theorem 4.4. Just observe that denoting by u the weak limit of u_n, the inequality

$$\mathcal{F}_g^+(u) \leq \lim_{n\to+\infty} \mathcal{F}_g^+(u_n)$$

follows by working separately on the intervals $[0, T]$ and $]T, +\infty[$ using also (4.23). $\square$

4.3. Multi-transition Homoclinics

Homoclinic solutions of (4.6) to ± 1 belong respectively to the functional spaces

$$1 + H^2(\mathbb{R}) \quad \text{and} \quad -1 + H^2(\mathbb{R}).$$

We choose to focus on homoclinics to $+1$. When minimizing the functional $\mathcal{F}_g$ defined by (4.5) in $1+H^2(\mathbb{R})$, it is natural to search for even homoclinic solutions. Indeed, suppose that $u-1 \in H^2(\mathbb{R})$, then there exists an even function u^* that satisfies $u^* - 1 \in H^2(\mathbb{R})$ and $\mathcal{F}_g^+(u^*) \leq \mathcal{F}_g^+(u)$. Observe that if $\bar{x}$ is a critical point of u, writing $J_1 = (-\infty, \bar{x}]$ and $J_2 = (\bar{x}, +\infty)$, the action of u is smaller in either J_1 or J_2. Assuming that u has a lower action in J_1, we define $u^* \in 1 + H^2(\mathbb{R})$ by

$$u^* = \begin{cases} u(x) & \text{if } x \in J_1, \\ u(2\bar{x} - x) & \text{if } x \in J_2. \end{cases}$$

Since the Lagrangian is reversible, we then conclude that

$$\mathcal{F}_g^+(u^*) = 2\mathcal{F}_g|_{J_1}(u) \leq \mathcal{F}_g|_{J_1}(u) + \mathcal{F}_g|_{J_2}(u) = \mathcal{F}_g^+(u)$$

and by translation invariance, we may also assume that $\bar{x} = 0$.

We therefore define the functional space

$$\tilde{\mathcal{E}}^+ := \{u \in C^1(\mathbb{R}^+) \mid u - 1 \in H^2(\mathbb{R}^+),\ u'(0) = 0\}. \tag{4.24}$$

It is easily seen that if $u \in \tilde{\mathcal{E}}^+$ is a critical point of the functional $\mathcal{F}_g^+$, then $u'''(0) = 0$ is a natural boundary condition. It then follows that the even extension of u to $\mathbb{R}$ is a solution of (4.6) which is at least C^4. Also, it is not difficult to verify that any critical point in $\tilde{\mathcal{E}}^+$ satisfies the conditions

$$\lim_{x \to \pm\infty} (u(x), u'(x), u''(x), u'''(x)) = (+1, 0, 0, 0). \tag{4.25}$$

As in Section 4.2, we assume that f and g are even and the functional $\mathcal{F}_g^+$ is bounded from below in $\tilde{\mathcal{E}}^+$. Actually, when ± 1 are saddle-focus equilibria, this again implies that $\mathcal{F}_g^+$ is non-negative in $\tilde{\mathcal{E}}^+$.

It is obvious that minimizing $\mathcal{F}_g^+$ in $\tilde{\mathcal{E}}^+$ leads to the trivial solution $u = 1$. Moreover, $u = 1$ is the only function having zero action if $\mathcal{F}_g^+$ is bounded from below. In order to get non-trivial solutions, we minimize $\mathcal{F}_g^+$ in subclasses $\tilde{\mathcal{E}}_p^+ \subset \tilde{\mathcal{E}}^+$ that do not contain the function $u = 1$. We define for each $p \geq 0$ the subset $\tilde{\mathcal{E}}_p^+ \subset \tilde{\mathcal{E}}^+$ consisting of functions whose even extensions to $\mathbb{R}$ make $2p$ transitions. Precisely, we introduce the following definition.

Definition 4.15. *A function* $u \in \tilde{\mathcal{E}}^+$ *belongs to the subclass* $\tilde{\mathcal{E}}_p^+$ *if there exist* $0 = x_0 < x_1 < \ldots < x_p < x_{p+1} = +\infty$ *such that*

$$u(x)(-1)^{i+p} > 0 \text{ for } x \in I_i$$

and

$$\max_{x \in I_i} u(x)(-1)^{i+p} > 1,$$

where I_i *denote the interval* $]x_i, x_{i+1}[$ *for every* $i = 0, \ldots, p$.

Arguing as in Section 4.2.1 and Section 4.2.2, we are able to prove that $\mathcal{F}_g^+$ has a local minimum in each of these subspaces in the two following situations.

Theorem 4.16. *Let* f, $g \in C^2(\mathbb{R})$ *be even functions such that* (C2) *holds and* $f(1) = 0$. *Assume further that* $g(1)^2 < 4f''(1)$ *and*

$$\inf_{\tilde{\mathcal{E}}^+} \mathcal{F}_g^+ > -\infty,$$

where the functional $\mathcal{F}_g^+ : \tilde{\mathcal{E}}^+ \to \mathbb{R}$ *is defined by* (4.7) *and* (4.24). *Then the functional* $\mathcal{F}_g^+$ *has a local minimizer in each subspace* $\tilde{\mathcal{E}}_p^+$. *Moreover, its even extension to* $\mathbb{R}$ *is a homoclinic solution of* (4.6) *to* $+1$, *which has exactly* $2p$ *zeros.*

Theorem 4.17. *Let* $f \in C^2(\mathbb{R})$ *be a non-negative even function that satisfies assumptions* (B1) *and* (B3). *Assume* $g \in C^2(\mathbb{R})$ *is an even function that satisfies* $g(1)^2 < 4f''(1)$ *and assumption* (C1). *Then the conclusion of Theorem* 4.16 *holds.*

The proofs of these results are basically identical to these of Theorem 4.4 and Theorem 4.13. Observe that in order to adapt the second step of the proof of Theorem 4.4, we need the equivalent of Lemma 4.11 for functions $u \in H^2(a, b)$ that satisfy $u'(a) = u'(b) = 0$. Also, some other arguments are slightly different, taking into account that the boundary condition $u'(0) = 0$ substitute the condition $u(0) = 0$ for functions $u \in \tilde{\mathcal{E}}^+$.

Under the assumptions of Theorem 4.4 and Theorem 4.16, it is easy to check that $\inf_{\mathcal{E}^+} \mathcal{F}_g^+ > -\infty$ if and only if $\inf_{\tilde{\mathcal{E}}^+} \mathcal{F}_g^+ > -\infty$. Therefore, we obtain homoclinic solutions for the Swift-Hohenberg equation for the same range of the parameter β as in Proposition 4.12.

Proposition 4.18. *Let* β_0 *be defined by* (4.18). *Then, if* $\beta \geq \beta_0$, *for all* $p \in \mathbb{N}$, *the functional* $\mathcal{F}_\beta^+ : \tilde{\mathcal{E}}^+ \to \mathbb{R}$ *defined by* (4.19) *and* (4.24), *has a local minimizer* $\tilde{u}_p \in \tilde{\mathcal{E}}_p^+$ *whose even extension is a solution of* (4.1) *homoclinic to* $+1$ *having exactly* $2p$ *zeros. Moreover, for all* $p \in \mathbb{N}$, *we have* $\mathcal{F}_\beta^+(\tilde{u}_p) < \mathcal{F}_\beta^+(\tilde{u}_{p+1})$.

4.4. Notes and Comments

Note 4.1. The result of Section 4.1 holds in a more general framework. The potential

$$f(u) = \frac{(u^2 - 1)^2}{4}$$

may be replaced by a function $V \in C^2(\mathbb{R})$ that has exactly two nondegenerate global minima at $u = \pm 1$ and grows superquadratically at $\pm\infty$. If the parameter β satisfies $\beta^2 < 4V''(\pm 1)$ then $u = \pm 1$ are saddle-focus equilibria for the equation

$$u'''' - \beta u'' + V'(u) = 0.$$

If V is even, then the equivalent of Theorem 4.2 holds, while without this symmetry assumption it is not clear that the infima are attained in the interior of each homotopy class. However, when $g_i = 2$ for all i or if the g_i's are large enough (in this case, the profile is similar to that of a multi-bump solution), the local minimizers exist in the corresponding class $M(g)$, see W. D. Kalies et al. [**54**].

The equivalent of Theorem 4.2 for homoclinic connections was also considered in the same paper [**54**]. The methods of [**54**] were further adapted by the same authors and J. B. VandenBerg [**53**] to study homotopy classes of periodic and chaotic solutions.

Note 4.2. Theorem 4.4 can be adapted to the setting of Theorem 3.10, with an additional symmetry condition on f and g, to the cost of a more tedious proof.

Note 4.3. A remark similar to that in Note 3.3 may be formulated. Indeed, for $-\sqrt{2} < \beta \leq \beta_0$, the functional $\mathcal{F}_\beta^+$ is bounded from below in each class $\mathcal{E}_p^+$. Using an argument of continuity similar to that exposed in Note 3.3, we are able to prove that the local minimizers lie in the interior of the classes for β in a small left neighbourhood of β_0.

Note 4.4. The abundance of local minima suggest the existence of infinitely many other critical points of minimax type. However the lack of a compactness property of the Palais-Smale sequences makes the investigation of such solution very delicate. It would seem rather natural that a solution obtained from a minimax principle based on deformations from one class $\mathcal{E}_p^+$ to the following class $\mathcal{E}_{p+1}^+$, behaves like a single-transition solution of equation (4.1) obtained, for $\beta \geq 0$, by L. A. Peletier and W. C. Troy [**79**] using shooting arguments. Even for this range of β, the variational nature of the single-transition solutions is unknown.

For each $p \geq 0$ and for $\beta_0 \leq \beta$, the functional $\mathcal{F}_\beta^+$ has a local minimizer u_p in the interior of $\mathcal{E}_p^+$, see Proposition 4.12. Arguing as in

Note 3.3, we are able to prove that

$$\mathcal{F}_\beta^+(u_p) < \inf_{\partial\mathcal{E}_p^+} \mathcal{F}_\beta^+.$$

We now define the minimax levels

$$c_p := \inf_{\gamma\in\Gamma_p} \max_{t\in[0,1]} \mathcal{F}_\beta^+(\gamma(t))$$

where

$$\Gamma_p = \{\gamma \in \mathcal{C}([0,1],\mathcal{E}^+) \mid \gamma(0) = u_p,\ \gamma(1) = u_{p+1}\}.$$

From the positivity of $\mathcal{F}_{\beta_0}^+$ in $\mathcal{E}^+$, we infer that for $\beta > \beta_0$, every Palais-Smale sequence $(u_n)_n \subset \mathcal{E}^+$ at level c_p is such that $(u_n')_n$ is bounded in $H^1(\mathbb{R}^+)$. Consequently, any Palais-Smale sequence $(u_n)_n$ converges (up to a subsequence) to some function $u \in H^2_{\text{loc}}(\mathbb{R}^+)$ that has limit at infinity equal to either $+1$ or -1 (essentially because the potential energy is finite). If $u(+\infty) = +1$, the odd extension of u is a heteroclinic solution from -1 to $+1$. Otherwise, $-u \in \mathcal{E}^+$ and the odd extension of $-u$ is a heteroclinic solution connecting -1 to $+1$. However, it seems rather complicate to obtain information on the shape of these solutions and estimates on their action (which could be lower than the critical levels c_p and even equal to that of u_p). It would be interesting to check that these a priori "new" solutions do not coincide with the local minimizers.

NOTE 4.5. *Homoclinics for various fourth order models.*
Existence and multiplicity results of homoclinics for many other fourth order models were investigated in the literature. We list some of them:

(i) The Swift-Hohenberg equation. D. Smets and J. B.van den Berg [**106**] used a modified functional and mountain-pass arguments to catch a homoclinic solution of the Swift-Hohenberg equation for almost every $\beta \in]-\sqrt{8}, 0[$.

(ii) Suspension bridge models. In series of contributions [**30, 31, 33, 59, 69, 70**], P. J. McKenna et al. considered some models for travelling waves in suspension bridges. One of those is based on the beam equation with a nonlinear source term accounting for the external forces exerted on the bridge. These considerations lead to the model equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + (u+1)_+ - 1 = 0.$$

A travelling wave solution $u(x,t) = w(x-ct)$ then solves the ordinary differential equation

$$w'''' + c^2 w'' + (w+1)_+ - 1 = 0.$$

In this model, the potential $V : \mathbb{R} \to \mathbb{R}$ defined by

$$V(w) = \begin{cases} -w - \frac{1}{2} & \text{if } w \le -1, \\ \dfrac{w^2}{2} & \text{if } w > -1, \end{cases}$$

has a single-well located at $w = 0$. Therefore, the question of interest is that of the existence of a homoclinic solution to 0. In Y. Chen and P. J. McKenna [**33**], the existence of homoclinics to 0 for

$$w'''' + c^2 w'' + V'(w) = 0 \tag{4.26}$$

was proved under the assumptions that $V \in C^2(\mathbb{R})$ is a potential such that

$$V'(w) = (w+1)^+ - 1 + g(w),$$

with $g(0) = g'(0) = 0$, $ug''(u) \leq 0$ for all $u \neq 0$ and $|g''(u)| \leq K$ for some $K > 0$ and all $u \neq 0$. This result was improved by D. Smets and J. B. van den Berg [**106**]. They obtained homoclinic solutions to 0 for equation (4.26) for almost every $c \in [-\sqrt[4]{4\alpha}, \sqrt[4]{4\alpha}]$, assuming that

$$V(0) = V'(0) = 0, \ V''(0) = \alpha > 0$$

and

$$\limsup_{w \to -\infty} \frac{V(w)}{|w|^2} = 0.$$

(iii) Water waves. In the theory of shallow water waves driven by gravity and capillarity [**25**], a reduction of the full system of partial differential equations modelling the motion of the wave, lead to the single fourth-order ordinary differential equation

$$u'''' + Pu'' + u - u^2 = 0.$$

The parameter P is a negative constant and u is related to the depth of the water. This last equation was extensively considered by B. Buffoni et al. [**12, 21, 22, 23, 24, 27**]. Under some restriction on the parameter P, the constant solution $u = 0$ is a saddle-focus equilibrium. In the quoted papers, the set of periodic solutions and homoclinics to 0 is studied through the use of various methods. For instance, B. Buffoni [**22**] proved the existence of homoclinic solutions via variational arguments. Namely, he applied the Mountain Pass theorem to the corresponding functional and concentration compactness arguments of P. L. Lions [**63, 64**] to overcome the lack of a Palais-Smale condition.

Chapter 5

A Ginzburg-Landau Model for Ternary Mixtures: Connections Between Non-consecutive Equilibria

In this last chapter, we consider the equation

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0$$

with a triple-well potential f. As emphasized in the introduction, a scalar second order differential equations cannot possess heteroclinic connections between non-consecutive minima of the potential as it would contradict the law of energy conservation. We show in this chapter that when the middle-equilibrium is of saddle-focus type, the dynamics of the above fourth order equation display such connecting orbits. This problem was brought to our attention by H. Leitão [**61, 62**]. It appears in the context of phase transition phenomena in ternary mixtures with *amphiphilic molecules*. In Section 5.1, we briefly describe the corresponding Ginzburg-Landau model introduced by physicists and chemists. We then turn to the mathematical treatment of the problem. In Section 5.2, we prove the existence of a connection between the extremal equilibria of the above *three stable-states* equation. The results of this section are based on those in D. Bonheure, L. Sanchez, M. Tarallo and S. Terracini [**18**].

5.1. A Ginzburg-Landau Model for Ternary Mixtures

5.1.1. Binary Fluids. A good example of phase transition phenomena is provided by the mixing-demixing transitions of a fluid. It is well known that oil and water do not mix so that a binary fluid composed of oil and water has two separated homogeneous phases usually called oil-rich and water-rich phases. Assuming that the essence of the transitions can be described in terms of the concentration difference between oil and water, the model is basically based on a scalar order function which locally mesures this difference.

Consider a one-dimensional model in $\mathbb{R}$ and denote the position by x. Then we denote by $u(x)$ the measure of the concentration of water and oil at point x. To fix the ideas, we assume that $u = -1$ is the water-rich equilibrium state while $u = +1$ is the oil-rich equilibrium state. The admissible profiles of the mixture are then stationary points of a free energy functional (see [**48**]) of the form

$$\mathcal{F}_{\text{bin}}(u) = \int_{\mathbb{R}} (g_0\, u'^2 + f(u) - \mu u)\, dx. \tag{5.1}$$

The function f is the free-energy density, μ is the chemical potential difference between oil and water and g_0 is a positive parameter. The free-energy density f is usually approximated by an even (due to the symmetry under the interchange of the two components) fourth order polynomial

$$f(u) = a(u^2 - 1)^2.$$

Thermodynamic stability of homogeneous phases requires that $a > 0$. Hence, the function $f(u) - \mu u$ has two local minima which correspond to the two phases water-rich and oil-rich. The parameter μ then decides which phase lies at the lowest energy level. At $\mu = 0$, the two phases are at the same level of energy.

A solution $u(x)$ of the Euler-Lagrange equation associated to the functional $\mathcal{F}_{\text{bin}}$ describe a time-independent configuration of the mixture. For example, a pulse profile which tends to -1 at both $\pm\infty$ and has only one jump to $+1$ represents a single drop of oil surrounded by water. The case where $\mu = 0$ in $\mathcal{F}_{\text{bin}}$ is of particular interest as it makes possible the formation of a separation of the water and the oil in the regions $x \to -\infty$ and $x \to +\infty$. A minimizer of the functional $\mathcal{F}_{\text{bin}}$ in the space of functions which spatially connect the two phases is then called an interfacial profile between the oil-rich and the water-rich phase at coexistence. Mathematically speaking, the minimizer is a heteroclinic solution of the underlying Euler-Lagrange equation connecting the two phases. In this context, a monotone profile connecting -1 to $+1$ indicates a fair separation of the water and the oil while a multi-transition profile means some drops of oil are scattered in an intermediate region between the water and the oil which extend respectively to $-\infty$ and $+\infty$.

5.1.2. Ternary Mixtures. The addition of an *amphiphile*[1] into the fluid may induce the formation of a third phase which contains

[1] an amphiphilic molecule consists of a polarizable head which prefers the water environment and a hydrocarbon tail which prefers the oil so that it may either mix with water or oil.

more of the amphiphile and less of water and oil than the other two. Its density will be intermediate between that of the water-rich and the oil-rich phases and will be therefore physically located between them. Hence, we call it the middle-phase and we assume it is located at $u = 0$.

In order to keep a scalar function u, which still mesures the density difference between oil and water, to describe the oil-water-amphiphile mixtures, we need to consider three-phase coexistence at $\mu = 0$ so that the potential f in (5.1) must have three minima at -1, 0 and $+1$ corresponding respectively to the phases oil-rich, middle and water-rich. Choosing the minimum level of the potential to be zero, the minimum profile $u(x)$ of $\mathcal{F}_{\text{bin}}$ at $\mu = 0$, between the oil-rich phase which extends to $x \to -\infty$ and the water-rich phase which extends to $x \to +\infty$, then should solve the energy conservation equation

$$g_0 u'(x)^2 - f(u(x)) = 0. \tag{5.2}$$

A simple phase-plane analysis, see Figure 9 in the introduction, shows that the only trajectory starting form -1 as $x \to -\infty$ and going to $+1$ as $x \to +\infty$ spends an infinite amount of time in the middle-phase. Hence, the functional $\mathcal{F}_{\text{bin}}$ has no minimizer u satisfying $u(-\infty) = -1$ and $u(+\infty) = +1$. In the ternary mixture problem, this means that the model predicts the middle phase will always wet the interface between the oil- and water-rich phases, a prediction contrary to the results of experiment.

A simple way proposed by G. Gompper and M. Schick [**48**] to overcome this consequence of the model and yet consider a single-order theory is to add a second order term in the Lagrangian, considering therefore the functional

$$\mathcal{F}_{\text{ter}}(u) = \int_{\mathbb{R}} \left(cu''^2 + g(u)u'^2 + f(u) - \mu u \right) \, dx.$$

The function g, which quantifies the properties of the amphiphile, is negative close to the middle-phase as it tends to create interfaces and positive in the oil and water phases. The parameter c is positive and stabilizes the system.

The aim of this chapter is to prove that this last functional does not suffer the defect of the classical Ginzburg-Landau model (5.1) at least under some hypotheses on the nature of the middle-phase equilibrium. Namely, we prove in Section 5.2 that the functional $\mathcal{F}_{\text{ter}}$ may possess a minimal profile connecting the extremal phases.

5.2. The Fourth Order Model

We consider a potential $f \in C^1(\mathbb{R})$ and a function $g \in C^2(\mathbb{R})$ satisfying assumption (C1) introduced in Chapter 3, i.e. there exist $\tilde{g} \in C(\mathbb{R})$ and some $k<1$ such that for all $u\in\mathbb{R}$,

$$g(u)\geq\tilde{g}(u)\quad\text{and}\quad|\tilde{G}(u)|\leq k\sqrt{8f(u)},$$

where $\tilde{G}(u):=\displaystyle\int_0^u\tilde{g}(s)\,ds$.

We also assume that

(D1) $f(u)=0$ if and only if $u=0$ or $u=\pm1$,

(D2) for some $0<a<1/2$ and $\alpha>0$,

$$\frac{f(u)}{(u-1)^2}\leq\alpha,\ \text{for } |u-1|<a,$$
$$\frac{f(u)}{(u+1)^2}\leq\alpha,\ \text{for } |u+1|<a,$$

(D3) $f(u)\geq 0$ for all $u\in\mathbb{R}$ and

$$\liminf_{|u|\to+\infty}f(u)>0,$$

(D4) f is of class C^2 close to 0, $f''(0)\neq 0$ and

$$g(0)^2<4f''(0).$$

It is by now familiar that assumption (D4) implies the trivial solution is a saddle-focus equilibrium for the linear equation

$$u''''-g(0)u''+f''(0)u=0.$$

As observed in Section 3.1, if $g(0)<0$ or (C1) holds with $g=\tilde{g}$, we just need f to be C^2 in a neighborhood of 0 and $f''(0)\neq 0$ as the inequality $g(0)^2<4f''(0)$ then automatically holds.

Under the previous assumptions, it makes sense to consider the functional

$$\mathcal{F}_g(u)=\int_{\mathbb{R}}\left(\frac{1}{2}(u''^2+g(u)u'^2)+f(u)\right)dx \tag{5.3}$$

in the space

$$\mathcal{H}=\{u\in C^1(\mathbb{R})\mid u''\in L^2(\mathbb{R}),\ u'\in L^\infty(\mathbb{R}),\ \lim_{x\to\pm\infty}u(x)=\pm1\}. \tag{5.4}$$

The corresponding Euler-Lagrange equation is given by

$$u''''-g(u)u''-\frac{1}{2}g'(u)u'^2+f'(u)=0. \tag{5.5}$$

Arguing as in Lemma 3.1, we observe that the functional (5.3) is bounded from below in $\mathcal{H}$.

The main theorem of the section goes as follows.

THEOREM 5.1. *Suppose that* $f \in C^1(\mathbb{R})$, $g \in C^2(\mathbb{R})$ *satisfy assumptions* (C1), (D1), (D2), (D3) *and* (D4). *Then, there exists a minimizer of the functional* $\mathcal{F}_g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *defined by* (5.3) *and* (5.4), *which is a weak heteroclinic solution of* (5.5) *connecting* -1 *to* $+1$.

In comparison with Theorem 2.8 and Theorem 3.3, the additional condition $g(0)^2 < 4f''(0)$ allows to consider a potential f with a third bottom at 0. We do not know if Theorem 5.1 holds without this assumption.

As for Theorem 2.8 and Theorem 3.3, the proof of Theorem 5.1 relies on the control and the relocalization of a minimizing sequence. We thus search for a control on the time it takes for a quasi-minimizer to travel from a neighborhood of $(-1,0)$ to a neighborhood of $(+1,0)$ in the uu'-plane. Observe that the presence of the third zero of the potential rules out some of the arguments used in the proof of Theorem 3.3. On the other hand, we easily obtain a control on time intervals from $(-1,0)$ to $(0,0)$ and $(0,0)$ to $(+1,0)$. To complete the arguments, we just need an estimate on the time quasi-minimizers spend close to 0. The latter is obtained thanks to the additional condition that 0 is a saddle-focus. As already observed in Section 3.3.1, the local minimizers of (5.3) close to 0 (in the phase space) change sign in every interval of length larger than a fixed constant. The clipping procedure introduced in Section 2.3.1 then implies that we may choose quasi-minimizers which do not spend much time close to 0. This idea was already used in the proofs of Theorem 3.10 and Theorem 4.4.

PROPOSITION 5.2. *Suppose that* $f \in C^1(\mathbb{R})$, $g \in C^2(\mathbb{R})$ *satisfy* (C1), (D1), (D2), (D3) *and* (D4). *Then, there exists* $L > 0$, $T > 0$ *and a minimizing sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that for all* $n \in \mathbb{N}$,
(i) $\|u_n\|_{C^1} \leq L$,
(ii) $|u_n(x) + 1| \leq a$ *for all* $x \leq -T$ *and* $|u_n(x) - 1| \leq a$ *for all* $x \geq T$.

PROOF. PART 1 - Property (i). The proof of the first statement has been worked out in the proof of Proposition 2.9 and adapted to the current framework in the proof of Theorem 3.3.

PART 2 - Property (ii). We divide the proof of the second statement in three main steps. In Step 1, we approximate the minimum of $\mathcal{F}_g$ with functions that stay close to ± 1 at $\pm\infty$. Step 2 estimates the time for a function $u \in \mathcal{H}$ to travel in the (u, u')-plane from neighborhoods of $(\pm 1, 0)$ to a neighborhood of $(0, 0)$. Finally, in Step 3, we show that functions in $\mathcal{H}$ that stay close to 0 with small velocity can be locally replaced, using the clipping procedure, by functions that spend in such a neighborhood a time which is a priori bounded.

Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence for $\mathcal{F}_g$. Let $\varepsilon > 0$ be fixed and for each $n \in \mathbb{N}$, define

$$x_1 := \sup\{x \mid |u_n(x) + 1| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\},$$

and

$$x_4 := \inf\{x \mid |u_n(x) - 1| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\}.$$

Step 1 - Modification of u_n in $]-\infty, x_1]$ and $[x_4, +\infty[$. Arguing as in the proof of Theorem 3.3, we may assume that for all $n \in \mathbb{N}$, $|u_n(x)+1| \leq a$ for all $x \leq x_1$ and $|u_n(x) - 1| \leq a$ for all $x \geq x_4$.

Step 2 - Estimates on time intervals. Define for all $n \in \mathbb{N}$,

$$x_2 := \inf\{x \geq x_1 \mid |u_n(x)| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\}$$

and

$$x_3 := \sup\{x \leq x_4 \mid |u_n(x)| \leq \varepsilon \text{ and } |u_n'(x)| \leq \varepsilon\}.$$

Notice that x_2 and x_3 need not exist. Arguing as in the proof of Proposition 2.9, we obtain a uniform bound for $x_2 - x_1$ and $x_4 - x_3$, or for $x_4 - x_1$ if x_2 and x_3 do not exist.

Step 3 - Modification in $[x_2, x_3]$. If x_2 and x_3 do not exist, this step can obviously be skipped.

Claim - There exists a positive constant τ_0 such that for every $n \in \mathbb{N}$, if $x_3 - x_2 \geq \max(1, 8\tau_0)$, the function u_n may be replaced by a clip $\hat{u}_n \in \mathcal{H}$ so that after clipping the interval $[x_1, x_4]$ is transformed into an interval of length smaller than $(x_2 - x_1) + \max(1, 8\tau_0) + (x_4 - x_3)$. Further, we have $\mathcal{F}_g(\hat{u}_n) \leq \mathcal{F}_g(u_n)$. We may assume u_n is a minimizer of $\mathcal{F}_g$ in the space

$$\mathcal{H}_{[x_2,x_3]} := \{w \in \mathcal{H} \mid w = u_n \text{ on } \mathbb{R} \setminus [x_2, x_3]\}.$$

It follows from an analysis similar to that of Lemma 3.13 that for any $\delta > 0$, there exist $\tau_0 > 0$ and $\varepsilon_0 > 0$ such that if $x_3 - x_2 \geq 1$ and $0 < \varepsilon \leq \varepsilon_0$, any minimizer u of $\mathcal{F}_g$ in $\mathcal{H}_{[x_2,x_3]}$ satisfy

$$\|u\|_{C^3([x_2,x_3])} \leq \delta$$

and changes sign in each interval of length greater than τ_0. Observe we do not assume that functions of $\mathcal{H}_{[x_2,x_3]}$ are small in the interval $[x_2, x_3]$. Here, this is a consequence of the smallness of the boundary conditions.

We then define

$$x_2' := \max\{x \leq x_2 \mid |u_n(x)| = \delta\} \quad \text{and} \quad x_3' := \min\{x \geq x_3 \mid |u_n(x)| = \delta\},$$

so that $|u_n(x)| \leq \delta$ for all $x \in [x_2', x_3']$. Suppose first that $u_n(x_2') = -\delta$ and $u_n(x_3') = \delta$. As usual, we define

$$s_2 := \min\{x \in [x_2', x_3'] \mid u_n'(x) = 0 \text{ and } u_n(x) \geq 0\},$$
$$s_4 := \max\{x \in [x_2', x_3'] \mid u_n(x) = u_n(s_2)\},$$
$$s_3 := \max\{x \in [s_2, s_4] \mid u_n'(x) = 0\}$$

and take

$$s_1 := \max\{x \in [x_2', s_2] \mid u_n(x) = u_n(s_3)\}.$$

Observe that $s_2 \in [x_2', x_2 + 2\tau_0]$ and $s_3 \in [x_3 - 2\tau_0, x_3']$. Further, we can apply the clipping procedure to the function u_n in the interval $[s_1, s_4]$ and discard the restriction of u_n to some interval $[\alpha, \beta]$ containing $[s_2, s_3]$. Letting $x_4^* := x_4 - (\beta - \alpha)$, we have

$$x_4^* - x_1 \leq (x_4 - x_3) + 4\tau_0 + (x_2 - x_1).$$

The case where $u_n(x_2') = \delta_0$ and $u_n(x_3') = -\delta_0$ is handled in the same way.

If $u_n(x_2') = -\delta_0$ and $u_n(x_3') = -\delta_0$ or $u_n(x_2') = \delta_0$ and $u_n(x_3') = \delta_0$, we proceed in two steps. Here, denoting by x_4^{**} the point into which x_4 is transformed after clipping, we obtain $x_4^{**} - x_1 \leq (x_4 - x_3) + 8\tau_0 + (x_2 - x_1)$.

Conclusion - It follows from the preceding steps that for some $T > 0$, we may assume that for each $n \in \mathbb{N}$, u_n satisfies property (ii). □

Now that we have at hand a minimizing sequence satisfying Proposition 5.2, the proof of Theorem 5.1 follows the lines of the proof of Theorem 5.1.

5.3. Notes and Comments

NOTE 5.1. As already pointed out, to our knowledge, it is not known whether Theorem 5.1 still holds without the assumption that 0 is a saddle-focus equilibrium. Numerical experiments seem to be in favor of a positive answer. On the other hand, as proved by J. B. van den Berg [**114**], the bounded solutions of the Extended Fisher-Kolmogorov equation

$$u'''' - \beta u'' + u^3 - u = 0$$

for $\beta \geq \sqrt{8}$ behave like the bounded solutions of the Fisher equation

$$-u'' + u^3 - u = 0.$$

If the same is true for a fourth order equation with a triple-well potential such as $f(u) = (u^2 - 1)^2 u^2$, this would mean that in general, a solution starting from -1 at $-\infty$ cannot pass throughout the middle equilibrium.

NOTE 5.2. Multi-transition solutions can also be obtained in the case of triple-well potentials, assuming that every equilibria are saddle-foci. We could even consider multiple-well potentials.

NOTE 5.3. We have seen in Section 1.4 that when a conservative Hamiltonian system of second order differential equations has more than two equilibria at the minimum level of the potential, we cannot ensure the existence of heteroclinic connections between each pair of equilibria but only the existence of chains of heteroclinics. It could be an interesting question to ask under which assumption on the nature of the equilibria a system of fourth order differential equations displays connections between each pair of equilibria at the same minimum level of the potential. In the fourth order case, the result concerning the scalar equation is no longer an obstacle.

NOTE 5.4. Other Ginzburg-Landau models for ternary mixtures have been proposed in the literature, including two-order-parameter and three-order-parameter models, see [**48**]. In these models, a second scalar function v which describe the local concentration of amphiphile is introduced, leading to free-energy functionals of the form

$$(u,v) \to \int_{\mathbb{R}} \big(\alpha_1 u''^2 + \alpha_2 u'^2 + \beta_1 v'^2 + \gamma_1 u'' v^2 + \gamma_2 u'^2 v + f(u,v)\big)\, dx,$$

where α_1, α_2, β_1, γ_1, $\gamma_2 \in \mathbb{R}$ and $f(u,v)$ is the sum of three terms, one triple-well potential depending only on u, one single-well potential depending only on v and a last one accounting for the coupling of u and v. Simple cases have been studied numerically by physicists but as far as we know, the corresponding type of functional has not been treated mathematically.

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